摘要

1961 年， Piatetski-Shapiro[1] 定义了 Siegel 域并证明了任何 Siegel 域全纯同构于有界域．接着， Vinberg, Gindikin 和 Piatetski-Shapiro[2] 于 1963 年证明了任何齐性有界域全纯同构于齐性 Siegel 域． 1976 和 1977 年，许以超 [3] 构造了一类特殊的齐性 Siegel 域，即正规 Siegel 域 $D(V_N,F)$, 并且在 [4] 中证明了任何齐性 Siegel 域仿射等价于正规 Siegel 域． 1976 年，许以超还定出了全纯自同构群 $\mathrm{Aut}\,(D(V_N,F))$ 的李代数 $\mathrm{aut}\,(D(V_N,F))$ 和仿射自同构群 $\mathrm{Aff}\,(D(V_N,F))$ 的生成元．与此同时， Dorfmeister 给出了齐性有界域的一个代数实现和全纯自同构群，但是该工作中一些全纯自同构群的存在条件并不清楚．

用 Cauchy-Szegö 核 $S(z,\overline{\xi})$, 华罗庚 [6] 构造了典型域上的形式泊松核 $P(z,\overline{z},\xi,\overline{\xi}) = |S(z,\overline{\xi})|^2/S(z,\overline{z})$, 并证明了形式泊松核是泊松核函数． 1965 年， Koranyi 用李群理论证明了形式泊松核是对称 Siegel 域上的泊松核．若 D 是一个不可分解的正规 Siegel 域，许以超证明了形式泊松核是泊松核当且仅当 D 是对称 Siegel 域．注意到对称域的 Silov 边界 $S(D)$ 在迷向子群的作用下是可递边界，而对于非对称域的情形，在迷向子群的作用下， Silov 边界 $S(D)$ 不是可递边界，我们可以提出下述问题：当 D 是非对称齐性 Siegel 域，在 Bergman 度量下， Silov 边界 $S(D)$ 上的连续函数类到 Laplace–Beltrami 调和函数类的积分表示是什么？

为了考虑泊松积分，我们需要得到正规 Siegel 域最大连通全纯自同构群 $\mathrm{Aut}\,(D(V_N,F))$ 的生成元集和固定点 $(\sqrt{-1}v_0,0)$ 的迷向子群 $\mathrm{Iso}\,(D(V_N,F))$ 的生成元集的确切表示，进而给出在迷向子群 $\mathrm{Iso}\,(D(V_N,F))$ 作用下 Silov 边界所有轨道的确切表示．本文给出了固定点 $(\sqrt{-1}v_0,0)$ 的迷向子群 $\mathrm{Iso}\,(D(V_N,F))$ 的生成元集并加以证明．

本文共分三章，第一章简要介绍有关背景及本文的所要解决的问题；第二章给出本文所用的一些符号及后面证明所需要的一些定义和定理；第三章在许以超教授关于正规 Siegel 域所做工作的基础上，通过求解一些常微分方程组，给出一些单参数子群，从而得到了正规 Siegel 域的迷向子群的生成元集．

关键词： 正规 Siegel 域，全纯自同构群，迷向子群，单参数子群，齐性 Siegel 域，齐性有界域．

ABSTRACT

In 1961, Piatetski-Shapiro[1] defined Siegel domains and proved that any Siegel domain is holomorphically isomorphic to a bounded domain. Successively, in 1963, Vinberg, Gindikin and Piatetski-Shapiro[2] proved that any homogeneous bounded domain is holomorphically isomorphic to a homogeneous Siegel domain. In 1976 and 1977, Yichao Xu[3] constructed a class of special homogeneous Siegel domain, i. e. normal Siegel domain $D(V_N, F)$, and proved that any homogeneous Siegel domain is affine equivalent to a normal Siegel domain in [4]. Also in 1976, Xu determined the Lie algebra aut $(D(V_N, F))$ of the holomorphic automorphism group Aut $(D(V_N, F))$ and the generating elements of the affine automorphism group Aff $(D(V_N, F))$. At the same time, Dorfmeister gave an algebraic realization of a homogeneous bounded domain and the holomorphic automorphism group, but the existence condition of some holomorphic automorphisms in his work is not clear.

From Cauchy-Szegö kernel $S(z, \bar{\xi})$, Hua[6] constructed the formal Poisson kernel $P(z, \bar{z}, \xi, \bar{\xi}) = |S(z, \bar{\xi})|^2 / S(z, \bar{z})$ on a classical domain, and proved that the formal Poisson kernel is the Poisson kernel function. In 1965, Koranyi proved that the formal Poisson kernel is the Poisson kernel on the symmetric Siegel domains using the Lie group theory. When D is an indecomposable normal Siegel domain, Xu proved that a formal Poisson kernel is Poisson kernel if and only if D is a symmetric siegel domain. Note that the Silov boundary $S(D)$ of a symmetric domain is a transitive boundary acted by the isotropic subgroup, but in the case of non-symmetric domain, the Silov boundary $S(D)$ is not a transitive boundary acted by the isotropic subgroup. So we can pose the following problem: when D is a non-symmetric homogeneous Siegel domain, what is the integral representation from the continuous function class on Silov boundary $S(D)$ to the Laplace-Beltrami harmonic function class with respect to the Bergman metric on D?

In order to consider the Poisson integral, we need to give the explicit expressions of the following two sets: one is the generating elements of the maximal connected holomorphic automorphism group Aut $(D(V_N, F))$ acting on the normal Siegel domains, the other is the generating elements of the isotropic subgroup Iso $(D(V_N, F))$ of the fixed point $(\sqrt{-1}v_0, 0)$. Particularly, we give the explicit expression of all orbits in the Silov boundary under the action by the isotropic subgroup Iso $(D(V_N, F))$. In this thesis, we will give the generating element set of the isotropic subgroup Iso $(D(V_N, F))$ of the fixed point $(\sqrt{-1}v_0, 0)$.

The thesis consists of three chapters. Chapter one gives a brief introduction to the relevant background and poses the problem which we will solve in this thesis. Chapter two introduces some necessary definitions and theorems needed for further arguments. In chapter three, based on professor Yichao Xu's work, we obtain the main results of this thesis: give some one parameter subgroups by solving some ordinary differential equations and get the generating element set of isotropic subgroup of normal Siegel domain.

Key words: normal Siegel domain, holomorphic automorphism group, one parameter subgroup, homogeneous Siegel domain, isotropic subgroup, homogeneous bounded domain.

第一章　主要结果

1959 年，Piatetski-Shapiro 在研究对称有界域的自守函数时发现了 Siegel 域，并在 1961 年 [1] 给出了其严格定义，证明了任何 Siegel 域全纯同构于有界域. 接着，Piatetski-Shapiro 与 Vinberg, Gindikin 于 1963 年 [2] 证明了任何齐性有界域全纯同构于齐性 Siegel 域，从而将齐性有界域的分类问题归结为齐性 Siegel 域的分类问题.

1976 和 1977 年，许以超 [3] 构造了 $\mathbb{C}^n$ 中一类特殊的齐性 Siegel 域，即正规 Siegel 域 $D(V_N,F)$. 正规 Siegel 域是由满足一定条件的特殊矩阵集合 (N 矩阵组) 定义的，并且在 [4] 中证明了任何齐性 Siegel 域仿射等价于正规 Siegel 域. 这样齐性 Siegel 域的分类就转化为正规 Siegel 域的分类，进而转化为 N 矩阵组在等价意义下的分类. 但是这个分类问题至今还未被完全解决. 1976 年，许以超还定出了正规 Siegel 域全纯自同构群 $\mathrm{Aut}\,(D(V_N,F))$ 的李代数 $\mathrm{aut}\,(D(V_N,F))$ 和仿射自同构群 $\mathrm{Aff}\,(D(V_N,F))$ 的生成元，并给出了 $D(V_N,F)$ 中一固定点 $p=(\sqrt{-1}v_0,0)$, $(v_0=(1,0,1,\cdots,0,1))$ 的迷向子群 $\mathrm{Iso}\,(D(V_N,F))$ 的李代数 $\mathrm{iso}\,(D(V_N,F))$. 与此同时，Dorfmeister 给出了齐性有界域的一个代数实现和全纯自同构群，但是该工作中一些全纯自同构群的存在条件并不清楚.

用 Cauchy-Szegö 核 $S(z,\bar{\xi})$，华罗庚 [6] 构造了典型域上的形式泊松核 $P(z,\bar{z},\xi,\bar{\xi})=|S(z,\bar{\xi})|^2/S(z,\bar{z})$，并证明了形式泊松核是泊松核函数. 1965 年，Koranyi 用李群理论证明了形式泊松核是对称 Siegel 域上的泊松核. 若 D 是一个不可分解的正规 Siegel 域，许以超证明了形式泊松核是泊松核当且仅当 D 是对称 Siegel 域. 注意到对称域的 Silov 边界 $S(D)$ 在迷向子群的作用下是可递边界，而对于非对称域的情形，在迷向子群的作用下，Silov 边界 $S(D)$ 不是可递边界，我们可以提出下述问题：当 D 是非对称齐性 Siegel 域，在 Bergman 度量下，Silov 边界 $S(D)$ 上的连续函数类到 Laplace–Beltrami 调和函数类的积分表示是什么？

为了考虑泊松积分，我们需要得到正规 Siegel 域最大连通全纯自同构群 $\mathrm{Aut}\,(D(V_N,F))$ 的生成元集和固定点 $(\sqrt{-1}v_0,0)$ 的迷向子群 $\mathrm{Iso}\,(D(V_N,F))$ 的生成元集的确切表示，进而给出在迷向子群 $\mathrm{Iso}\,(D(V_N,F))$ 作用下 Silov 边界所有轨道的确切表示. 本文根据 exp 映射下李代数和李群的对应关系以及单参数子群与李群的李代数的密切联系，由李代数 $\mathrm{iso}_p(D(V_N,F))$ 确定出正规 Siegel 域 $D(V_N,F)$ 的迷向子群 $\mathrm{Iso}_p(D(V_N,F))$ 的生成元的集合.

注记 1: 本文从选题至行文到完成，都经许以超先生悉心指导，在此对许老师和师母吴老师在学业和生活上的关心和教导谨致衷心的感谢. 许，吴两位老师的敬业精神和做人品格将永远是我学习的榜样.

注记 2: 本文初步定稿后经王天泽教授在文字上作了些修改，在此对王老师几年来的帮助和关心表示感谢.

注记 3: 本文的写作参考了陈敏茹的硕士论文，谨此致谢.

下面叙述本文的主要结果：

定理 正规 Siegel 域 $D(V_N,F)$ 中一固定点 p 的迷向子群 $\mathrm{Iso}_p(D(V_N,F))$ 的生成元集可由下面的 (1)-(4) 给出：

(1) 李子群 $\exp(o(D(V_N,F))$:

$$
\begin{aligned}
&r_j = s_j, \quad 1 \le j \le N,\\
&w_{ij} = z_{ij}O_{ij}, \quad 1 \le i < j \le N,\\
&v_i = u_iU_i, \quad 1 \le i \le N,
\end{aligned}
$$

其中 $O_{ij} \in O(n_{ij})$, $U_i \in U(m_i)$, 且满足

$$
O'_{ik}A^{tk}_{ij}O_{jk} = \sum_{r=1}^{n_{ij}}(e_rO_{ij}e'_t)A^{rk}_{ij}, \quad \overline{U_i}'Q^{(t)}_{ij}U_j = \sum_{r=1}^{n_{ij}}(e_rO_{ij}e'_t)Q^{(r)}_{ij}.
$$

(2) $\exp(\tilde{g}(D(V_N,F)))$, 它由 $\exp(\theta(X^{(t)}_{ij} - Z^{(t)}_{ij}))$ 生成，其中 $\exp(\theta(X^{(t)}_{ij} - Z^{(t)}_{ij}))$ 为：

$$
\begin{aligned}
&r_p = s_p, \quad p \ne i,j,\\
&r_i = (\frac{1}{2}\cos 2\theta + \frac{1}{2})s_i + (-\frac{1}{2}\cos 2\theta + \frac{1}{2})s_j + \sin 2\theta z^{(t)}_{ij},\\
&r_j = (-\frac{1}{2}\cos 2\theta + \frac{1}{2})s_i + (\frac{1}{2}\cos 2\theta + \frac{1}{2})s_j - \sin 2\theta z^{(t)}_{ij},\\
&w^{(s)}_{ij} = z^{(s)}_{ij}, \quad s \ne t,\\
&w^{(t)}_{ij} = -\frac{1}{2}\sin 2\theta s_i + \frac{1}{2}\sin 2\theta s_j + \cos 2\theta z^{(t)}_{ij},\\
&w_{pi} = z_{pi}\cos\theta + z_{pj}\sum_s A^{sj}_{pi}e'_te_s\sin\theta, \quad p < i,\\
&w_{pj} = -z_{pi}\sum_s e'_se_t(A^{sj}_{pi})'\sin\theta + z_{pj}\cos\theta, \quad p < i,\\
&w_{ip} = z_{ip}\cos\theta + z_{pj}\sum_s (A^{sj}_{ip})'e'_te_s\sin\theta, \quad i < p < j,\\
&w_{pj} = -z_{ip}\sum_s e'_se_tA^{sj}_{ip}\sin\theta + z_{pj}\cos\theta, \quad i < p < j,\\
&w_{ip} = z_{ip}\cos\theta + z_{jp}(A^{tp}_{ij})'\sin\theta, \quad p > j,\\
&w_{jp} = -z_{ip}A^{tp}_{ij}\sin\theta + z_{jp}\cos\theta, \quad p > j,\\
&v_p = u_p, \quad p \ne i,j,\\
&v_i = u_i\cos\theta + u_j\overline{(Q^{(t)}_{ij})}'\sin\theta,\\
&v_j = -u_iQ^{(t)}_{ij}\sin\theta + u_j\cos\theta.
\end{aligned}
$$

(3) $\exp(\widetilde{L_1})$, 它由下面两类单参数子群生成.

(3-i) $\exp(\theta(Y^{(t)}_i - \tilde{P}^{(t)}_i))$:

$$r_i = -\frac{2}{f(\theta)} - \sqrt{-1},$$

$$v_i = \frac{\sqrt{-1}\sqrt{c-1}\cos(2\theta + c')e_t + c_2}{f(\theta)},$$

$$w_{pi} = \frac{1}{f(\theta)} \sum_s \frac{\sin(\sqrt{2\lambda_{pi}^{(s)}}\theta + c_{pi}^{(s)'})}{\sin c_{pi}^{(s)'}} f(0) z_{pi} U_{pi} e_s' e_s \overline{U}_{pi}', \quad p < i,$$

$$w_{ip} = \frac{1}{f(\theta)} \sum_r \frac{\sin(\sqrt{2\mu_{ip}^{(r)}}\theta + c_{ip}^{(r)'})}{\sin c_{ip}^{(r)'}} f(0) z_{ip} U_{ip} e_r' e_r \overline{U}_{ip}', \quad p > i,$$

$$r_p = \sum_{s,r} \frac{2\sqrt{-1} f^2(0) z_{pi} U_{pi} E_{ss} \overline{U}_{pi}' \overline{U}_{pi} E_{rr} U_{pi}' z_{pi}'}{\sin c_{pi}^{(s)'} \sin c_{pi}^{(r)'}}$$
$$\cdot \int (\frac{1}{f^2(\theta)} \int \sin(\sqrt{2\lambda_{pi}^{(s)}}\theta + c_{pi}^{(s)'}) \sin(\sqrt{2\lambda_{pi}^{(r)}}\theta + c_{pi}^{(r)'}) d\theta) d\theta, \quad p < i,$$

$$r_p = \sum_{s,r} \frac{2\sqrt{-1} f^2(0) z_{ip} U_{ip} E_{ss} \overline{U}_{ip}' \overline{U}_{ip} E_{rr} U_{ip}' z_{ip}'}{\sin c_{ip}^{(s)'} \sin c_{ip}^{(r)'}}$$
$$\cdot \int (\frac{1}{f^2(\theta)} \int \sin(\sqrt{2\mu_{ip}^{(s)}}\theta + c_{ip}^{(s)'}) \sin(\sqrt{2\mu_{ip}^{(r)}}\theta + c_{ip}^{(r)'}) d\theta) d\theta, \quad p > i,$$

$$v_p = \sum_{s,r} f(0) z_{pi} U_{pi} E_{rr} \overline{U}_{pi}' E_{st} \overline{(Q_{pi}^{(s)})}' \int \frac{\sin(\sqrt{2\lambda_{pi}^{(r)}}\theta + c_{pi}^{(r)'})}{\sin c_{pi}^{(r)'} f(\theta)} d\theta - \frac{1}{2} \sum_{s,r} f(0) z_{pi} U_{pi} E_{rr} \overline{U}_{pi}' e_s'$$
$$\cdot (\int \frac{\sqrt{2\lambda_{pi}^{(r)}} \cos(\sqrt{2\lambda_{pi}^{(r)}}\theta + c_{pi}^{(r)'}) f(\theta) - \sin(\sqrt{2\lambda_{pi}^{(r)}}\theta + c_{pi}^{(r)'}) f'(\theta)}{\sin c_{pi}^{(r)'} f^2(\theta)}$$
$$(\sqrt{-1}\sqrt{c-1}\cos(2\theta + c')e_t + c_2) d\theta) \overline{(Q_{pi}^{(s)})}', \quad p < i,$$

$$v_p = \sum_{s,r} f(0) z_{ip} U_{ip} E_{rr} \overline{U}_{ip}' E_{st} Q_{ip}^{(s)} \int \frac{\sin(\sqrt{2\mu_{ip}^{(r)}}\theta + c_{ip}^{(r)'})}{\sin c_{ip}^{(r)'} f(\theta)} d\theta - \frac{1}{2} \sum_{s,r} f(0) z_{ip} U_{ip} E_{rr} \overline{U}_{ip}' e_s'$$
$$\cdot (\int \frac{\sqrt{2\mu_{ip}^{(r)}} \cos(\sqrt{2\mu_{ip}^{(r)}}\theta + c_{ip}^{(r)'}) f(\theta) - \sin(\sqrt{2\mu_{ip}^{(r)}}\theta + c_{ip}^{(r)'}) f'(\theta)}{\sin c_{ip}^{(r)'} f^2(\theta)}$$
$$(\sqrt{-1}\sqrt{c-1}\cos(2\theta + c')e_t + c_2) d\theta) Q_{ip}^{(s)}, \quad p > i,$$

$$w_{lp} = \sum_{s,r,u,h} \frac{2\sqrt{-1} f^2(0) z_{pi} U_{pi} E_{rr} \overline{U}_{pi}' e_s' z_{li} U_{li} E_{hh} \overline{U}_{li}' A_{lp}^{ui} e_s' e_u}{\sin c_{pi}^{(r)'} \sin c_{li}^{(h)'}}$$
$$\cdot \int (\frac{1}{f^2(\theta)} \int \sin(\sqrt{2\lambda_{pi}^{(r)}}\theta + c_{pi}^{(r)'}) \sin(\sqrt{2\lambda_{li}^{(h)}}\theta + c_{li}^{(h)'}) d\theta) d\theta, \quad l < p < i,$$

$$w_{lp}=\sum_{s,r,u}\frac{2\sqrt{-1}f^2(0)z_{li}U_{li}E_{rr}\overline{U}_{li}{}'e_s'z_{ip}U_{ip}E_{uu}\overline{U}_{ip}{}'(A_{li}^{sp})'}{\sin c_{li^{(r)}}'\sin c_{ip}^{(u)'}}$$

$$\cdot\int(\frac{1}{f^2(\theta)}\int\sin(\sqrt{2\lambda_{li}^{(r)}}\theta+c_{li}^{(r)'})\sin(\sqrt{2\mu_{ip}^{(u)}}\theta+c_{ip}^{(u)'})d\theta)d\theta,\quad l<i<p,$$

$$w_{lp}=\sum_{s,r,u}\frac{2\sqrt{-1}f^2(0)z_{il}U_{il}E_{rr}\overline{U}_{il}{}'e_s'z_{ip}U_{ip}E_{uu}\overline{U}_{ip}{}'A_{il}^{sp}}{\sin c_{il}^{(r)'}\sin c_{ip}^{(u)'}}$$

$$\cdot\int(\frac{1}{f^2(\theta)}\int\sin(\sqrt{2\mu_{il}^{(r)}}\theta+c_{il}^{(r)'})\sin(\sqrt{2\mu_{ip}^{(u)}}\theta+c_{ip}^{(u)'})d\theta)d\theta,\quad i<l<p.$$

其中

$$c=\frac{4\sqrt{-1}s_i-4u_i^{(t)2}}{(s_i+\sqrt{-1})^2},\quad c'=\mathrm{arctg}\frac{\sqrt{-1}-s_i}{2u_i^{(t)}},\quad f(\theta)=\sqrt{c-1}\sin(2\theta+c')+\sqrt{-1};$$

在 v_i 的表达式中， $c_2=\frac{-2u_i+2u_i^{(t)}e_t}{s_i+\sqrt{-1}}$;

在 w_{pi} 的表达式中，酉矩阵 U_{pi} 使 Hermite 矩阵 $H=\sum_{s,r}e_s'e_t\overline{(Q_{pi}^{(s)})}'Q_{pi}^{(r)}e_t'e_r$ 对角化， $\lambda_{pi}^{(s)}$, $s=1,\cdots,n_{pi}$ 为 H 的特征值，

$$c_{pi}^{(s)'}=\mathrm{arctg}\frac{\sqrt{2\lambda_{pi}^{(s)}}f(0)z_{pi}U_{pi}e_s'}{(\sqrt{-1}f(0)(\sum_s(s_i+\sqrt{-1})u_pQ_{pi}^{(s)}e_t'e_s+\sum_{s,r}z_{pi}^{(s)}u_i\overline{(Q_{pi}^{(r)})}'Q_{pi}^{(s)}e_t'e_r)+f'(0)z_{pi})U_{pi}e_s'};$$

在 w_{ip} 的表达式中，酉矩阵 U_{ip} 使 Hermite 矩阵 $M=\sum_{s,r}e_s'e_tQ_{ip}^{(s)}\overline{(Q_{ip}^{(r)})}'e_t'e_r$ 对角化， $\mu_{ip}^{(r)},r=1,\cdots,n_{ip}$ 为 M 的特征值，

$$c_{ip}^{(r)'}=\mathrm{arctg}\frac{\sqrt{2\mu_{ip}^{(r)}}f(0)z_{ip}U_{ip}e_r'}{(\sqrt{-1}f(0)(\sum_s(s_i+\sqrt{-1})u_p\overline{(Q_{ip}^{(s)})}'e_t'e_s+\sum_{s,r}z_{ip}^{(s)}u_iQ_{ip}^{(r)}\overline{(Q_{ip}^{(s)})}'e_t'e_r)+f'(0)z_{ip})U_{ip}e_r'};$$

在 $r_p(p<i)$ 的表达式中,待定参数由初值 $r_p(0)=s_p$, $\frac{dr_p}{d\theta}(0)=2\sqrt{-1}\sum_s z_{pi}^{(s)}u_pQ_{pi}^{(s)}e_t'$ 确定；

在 $r_p(p>i)$ 的表达式中,待定参数由初值 $r_p(0)=s_p$, $\frac{dr_p}{d\theta}(0)=2\sqrt{-1}\sum_s z_{ip}^{(s)}u_p\overline{(Q_{ip}^{(s)})}'e_t'$ 确定；

在 $v_p(p<i)$ 的表达式中，待定参数由初值 $v_p(0)=u_p$ 确定；

在 $v_p(p<i)$ 的表达式中，待定参数由初值 $v_p(0)=u_p$ 确定；

在 $w_{lp}(l<p<i)$ 的表达式中，待定参数由初值 $w_{lp}(0)=z_{lp}$, $\frac{dw_{lp}}{d\theta}(0)=\sqrt{-1}\sum_{s,u}[u_pQ_{pi}^{(s)}e_t'z_{li}A_{lp}^{ui}e_s'e_u+u_lQ_{li}^{(s)}e_t'e_sA_{lp}^{ui}z_{pi}'e_u]$ 确定；

在 $w_{lp}(i<l<p)$ 的表达式中，待定参数由初值 $w_{lp}(0)=z_{lp}$, $\frac{dw_{lp}}{d\theta}(0)=\sqrt{-1}\sum_s u_l Q_{li}^{(s)} e_t' z_{ip}(A_{li}^{sp})' + \sqrt{-1}\sum_{s,u} z_{li}^{(u)} u_p \overline{(Q_{ip}^{(s)})}' e_t' e_s (A_{li}^{up})'$ 确定；

在 $w_{lp}(i<l<p)$ 的表达式中，待定参数由初值 $w_{lp}(0)=z_{lp}$, $\frac{dw_{lp}}{d\theta}(0)=\sqrt{-1}\sum_s u_l \overline{(Q_{il}^{(s)})}' e_t' z_{ip} A_{il}^{sp} + \sqrt{-1}\sum_{s,u} z_{il}^{(u)} u_p \overline{(Q_{ip}^{(s)})}' e_t' e_s A_{il}^{up}$ 确定.

(3-ii) $\exp(\theta(\widetilde{Y}_i^{(t)}+P_i^{(t)}))$:

$$r_i=-\frac{2}{f(\theta)}-\sqrt{-1},$$

$$v_i=\frac{-\sqrt{c-1}\cos(2\theta+c')e_t+c_2}{f(\theta)},$$

$$w_{pi}=\frac{1}{f(\theta)}\sum_s \frac{\sin(\sqrt{2\lambda_{pi}^{(s)}}\theta+c_{pi}^{(s)'})}{\sin c_{pi}^{(s)'}} f(0) z_{pi} U_{pi} e_s' e_s \overline{U}_{pi}{}', \quad p<i,$$

$$w_{ip}=\frac{1}{f(\theta)}\sum_r \frac{\sin(\sqrt{2\mu_{ip}^{(r)}}\theta+c_{ip}^{(r)'})}{\sin c_{ip}^{(r)'}} f(0) z_{ip} U_{ip} e_r' e_r \overline{U}_{ip}{}', \quad p>i,$$

$$r_p=\sum_{s,r}\frac{2\sqrt{-1}f^2(0) z_{pi} U_{pi} E_{ss} \overline{U}_{pi}{}' \overline{U}_{pi} E_{rr} U_{pi}' z_{pi}'}{\sin c_{pi}^{(s)'} \sin c_{pi}^{(r)'}}$$
$$\cdot\int\Big(\frac{1}{f^2(\theta)}\int \sin(\sqrt{2\lambda_{pi}^{(s)}}\theta+c_{pi}^{(s)'})\sin(\sqrt{2\lambda_{pi}^{(r)}}\theta+c_{pi}^{(r)'})d\theta\Big)d\theta, \quad p<i,$$

$$r_p=\sum_{s,r}\frac{2\sqrt{-1}f^2(0) z_{ip} U_{ip} E_{ss} \overline{U}_{ip}{}' \overline{U}_{ip} E_{rr} U_{ip}' z_{ip}'}{\sin c_{ip}^{(s)'} \sin c_{ip}^{(r)'}}$$
$$\cdot\int\Big(\frac{1}{f^2(\theta)}\int \sin(\sqrt{2\mu_{ip}^{(s)}}\theta+c_{ip}^{(s)'})\sin(\sqrt{2\mu_{ip}^{(r)}}\theta+c_{ip}^{(r)'})d\theta\Big)d\theta, \quad p>i,$$

$$v_p=\sqrt{-1}\sum_{s,r} f(0) z_{pi} U_{pi} E_{rr} \overline{U}_{pi}{}' E_{st} \overline{(Q_{pi}^{(s)})}' \int \frac{\sin(\sqrt{2\lambda_{pi}^{(r)}}\theta+c_{pi}^{(r)'})}{\sin c_{pi}^{(r)'} f(\theta)} d\theta - \sum_{s,r}\frac{f(0) z_{pi} U_{pi} E_{rr} \overline{U}_{pi}{}' e_s'}{2}$$
$$\cdot\Big(\int \frac{\sqrt{2\lambda_{pi}^{(r)}}\cos(\sqrt{2\lambda_{pi}^{(r)}}\theta+c_{pi}^{(r)'})f(\theta)-\sin(\sqrt{2\lambda_{pi}^{(r)}}\theta+c_{pi}^{(r)'})f'(\theta)}{\sin c_{pi}^{(r)'} f^2(\theta)}$$
$$(-\sqrt{c-1}\cos(2\theta+c')e_t+c_2)d\theta\Big)\overline{(Q_{pi}^{(s)})}', \quad p<i,$$

$$v_p=\sqrt{-1}\sum_{s,r} f(0) z_{ip} U_{ip} E_{rr} \overline{U}_{ip}{}' E_{st} Q_{ip}^{(s)} \int \frac{\sin(\sqrt{2\mu_{ip}^{(r)}}\theta+c_{ip}^{(r)'})}{\sin c_{ip}^{(r)'} f(\theta)} d\theta - \sum_{s,r}\frac{f(0) z_{ip} U_{ip} E_{rr} \overline{U}_{ip}{}' e_s'}{2}$$

$$\cdot (\int \frac{\sqrt{2\mu_{ip}^{(r)}}\cos(\sqrt{2\mu_{ip}^{(r)}}\theta + c_{ip}^{(r)'})f(\theta) - \sin(\sqrt{2\mu_{ip}^{(r)}}\theta + c_{ip}^{(r)'})f'(\theta)}{\sin c_{ip}^{(r)'} f^2(\theta)}$$

$$(-\sqrt{c-1}\cos(2\theta + c')e_t + c_2)d\theta)Q_{ip}^{(s)}, \quad p > i,$$

$$w_{lp} = \sum_{s,r,u,h} \frac{2\sqrt{-1}f^2(0)z_{pi}U_{pi}E_{rr}\overline{U}_{pi}{}'e_s'z_{li}U_{li}E_{hh}\overline{U}_{li}{}'A_{lp}^{ui}e_s'e_u}{\sin c_{pi}^{(r)'}\sin c_{li}^{(h)'}}$$

$$\cdot \int (\frac{1}{f^2(\theta)}\int \sin(\sqrt{2\lambda_{pi}^{(r)}}\theta + c_{pi}^{(r)'})\sin(\sqrt{2\lambda_{li}^{(h)}}\theta + c_{li}^{(h)'})d\theta)d\theta, \quad l < p < i,$$

$$w_{lp} = \sum_{s,r,u} \frac{2\sqrt{-1}f^2(0)z_{li}U_{li}E_{rr}\overline{U}_{li}{}'e_s'z_{ip}U_{ip}E_{uu}\overline{U}_{ip}{}'(A_{li}^{sp})'}{\sin c_{li}^{(r)'}\sin c_{ip}^{(u)'}}$$

$$\cdot \int (\frac{1}{f^2(\theta)}\int \sin(\sqrt{2\lambda_{li}^{(r)}}\theta + c_{li}^{(r)'})\sin(\sqrt{2\mu_{ip}^{(u)}}\theta + c_{ip}^{(u)'})d\theta)d\theta, \quad l < i < p,$$

$$w_{lp} = \sum_{s,r,u} \frac{2\sqrt{-1}f^2(0)z_{il}U_{il}E_{rr}\overline{U}_{il}{}'e_s'z_{ip}U_{ip}E_{uu}\overline{U}_{ip}{}'A_{il}^{sp}}{\sin c_{li}^{(r)'}\sin c_{ip}^{(u)'}}$$

$$\cdot \int (\frac{1}{f^2(\theta)}\int \sin(\sqrt{2\mu_{il}^{(r)}}\theta + c_{il}^{(r)'})\sin(\sqrt{2\mu_{ip}^{(u)}}\theta + c_{ip}^{(u)'})d\theta)d\theta, \quad i < l < p.$$

其中

$$c = \frac{4\sqrt{-1}s_i + 4{u_i^{(t)}}^2}{(s_i + \sqrt{-1})^2}, \quad c' = \text{arctg}\,\frac{1 + \sqrt{-1}s_i}{2u_i^{(t)}}, \quad f(\theta) = \sqrt{c-1}\sin(2\theta + c') + \sqrt{-1};$$

在 v_i 的表达式中， $c_2 = \frac{-2u_i + 2u_i^{(t)}e_t}{s_i + \sqrt{-1}}$;

在 w_{pi} 的表达式中，酉矩阵 U_{pi} 使 Hermite 矩阵 $H = \sum_{s,r} e_s'e_t\overline{(Q_{pi}^{(s)})'}Q_{pi}^{(r)}e_t'e_r$ 对角化， $\lambda_{pi}^{(s)}$, $s = 1, \cdots, n_{pi}$ 为 H 的特征值，

$$c_{pi}^{(s)'} = \text{arctg}\,\frac{\sqrt{2\lambda_{pi}^{(s)}}f(0)z_{pi}U_{pi}e_s'}{(f(0)(\sum_s(s_i + \sqrt{-1})u_pQ_{pi}^{(s)}e_t'e_s + \sum_{s,r} z_{pi}^{(s)}u_i\overline{(Q_{pi}^{(r)})'}Q_{pi}^{(s)}e_t'e_r) + f'(0)z_{pi})U_{pi}e_s'};$$

在 w_{ip} 的表达式中，酉矩阵 U_{ip} 使 Hermite 矩阵 $M = \sum_{s,r} e_s'e_tQ_{ip}^{(s)}\overline{(Q_{ip}^{(r)})'}e_t'e_r$ 对角化， $\mu_{ip}^{(r)}, r = 1, \cdots, n_{ip}$ 为 M 的特征值，

$$c_{ip}^{(r)'} = \text{arctg}\,\frac{\sqrt{2\mu_{ip}^{(r)}}f(0)z_{ip}U_{ip}e_r'}{(f(0)(\sum_s(s_i + \sqrt{-1})u_p\overline{(Q_{ip}^{(s)})'}e_t'e_s + \sum_{s,r} z_{ip}^{(s)}u_iQ_{ip}^{(r)}\overline{(Q_{ip}^{(s)})'}e_t'e_r) + f'(0)z_{ip})U_{ip}e_r'};$$

在 $r_p(p<i)$ 的表达式中，待定参数由初值 $r_p(0)=s_p$，$\frac{dr_p}{d\theta}(0)=2\sum\limits_s z_{pi}^{(s)}u_pQ_{pi}^{(s)}e_t'$ 确定；

在 $r_p(p>i)$ 的表达式中，待定参数由初值 $r_p(0)=s_p$，$\frac{dr_p}{d\theta}(0)=2\sum\limits_s z_{ip}^{(s)}u_p\overline{(Q_{ip}^{(s)})'}e_t'$ 确定；

在 $v_p(p<i)$ 的表达式中，待定参数由初值 $v_p(0)=u_p$ 确定；

在 $v_p(p<i)$ 的表达式中，待定参数由初值 $v_p(0)=u_p$ 确定；

在 $w_{lp}(l<p<i)$ 的表达式中，待定参数由初值 $w_{lp}(0)=z_{lp}$，$\frac{dw_{lp}}{d\theta}(0)=\sum\limits_{s,u}[u_pQ_{pi}^{(s)}e_t'z_{li}A_{lp}^{ui}e_s'e_u+u_lQ_{li}^{(s)}e_t'e_sA_{lp}^{ui}z_{pi}'e_u]$ 确定；

在 $w_{lp}(i<l<p)$ 的表达式中，待定参数由初值 $w_{lp}(0)=z_{lp}$，$\frac{dw_{lp}}{d\theta}(0)=\sum\limits_s u_lQ_{li}^{(s)}e_t'z_{ip}(A_{li}^{sp})'+\sum\limits_{s,u}z_{li}^{(u)}u_p\overline{(Q_{ip}^{(s)})'}e_t'e_s(A_{li}^{up})'$ 确定；

在 $w_{lp}(i<l<p)$ 的表达式中，待定参数由初值 $w_{lp}(0)=z_{lp}$，$\frac{dw_{lp}}{d\theta}(0)=\sum\limits_s u_l\overline{(Q_{il}^{(s)})'}e_t'z_{ip}A_{il}^{sp}+\sum\limits_{s,u}z_{il}^{(u)}u_p\overline{(Q_{ip}^{(s)})'}e_t'e_sA_{il}^{up}$ 确定.

(4) $\exp(\widetilde{L_2})$，它由下面两类单参数子群生成.

(4-i) $\exp(\operatorname{arctg}\theta(B_i+\frac{\partial}{\partial s_i}))$：

$$
\begin{aligned}
r_i &= \frac{s_i+\theta}{1-s_i\theta},\\
r_p &= s_p+\frac{\theta}{1-s_i\theta}z_{pi}z_{pi}', \quad p<i,\\
r_p &= s_p+\frac{\theta}{1-s_i\theta}z_{ip}z_{ip}', \quad p>i,\\
w_{pi} &= \frac{\sqrt{1+\theta^2}}{1-s_i\theta}z_{pi}, \quad p<i,\\
w_{ip} &= \frac{\sqrt{1+\theta^2}}{1-s_i\theta}z_{ip}, \quad p>i,\\
w_{kp} &= z_{kp}+\sum_s\frac{\theta}{1-s_i\theta}z_{ki}A_{kp}^{si}z_{pi}'e_s, \quad k<p<i,\\
w_{kp} &= z_{kp}+\sum_s\frac{\theta}{1-s_i\theta}z_{ki}^{(s)}z_{ip}(A_{ki}^{sp})', \quad k<i<p,\\
w_{kp} &= z_{kp}+\sum_s\frac{\theta}{1-s_i\theta}z_{ik}^{(s)}z_{ip}A_{ik}^{sp},\\
v_i &= \frac{\sqrt{1+\theta^2}}{1-s_i\theta}u_i,\\
v_p &= u_p+\sum_s\frac{\theta}{1-s_i\theta}z_{pi}^{(s)}u_i\overline{(Q_{pi}^{(s)})'}, \quad p<i,\\
v_p &= u_p+\sum_s\frac{\theta}{1-s_i\theta}z_{ip}^{(s)}u_iQ_{ip}^{(s)}, \quad p>i.
\end{aligned}
$$

(4-ii) $\exp\left(\theta(T_{ij}^{(t)} + 2\frac{\partial}{\partial z_{ij}^{(t)}})\right)$:

$$r_i = s_i \frac{\cos^2\sqrt{2}c' + A^2\sin^2\sqrt{2}c'}{\cos^2\sqrt{2}\varphi + A^2\sin^2\sqrt{2}\varphi},$$
$$r_j = s_j \frac{\cos^2\sqrt{2}c' + A^2\sin^2\sqrt{2}c'}{\cos^2\sqrt{2}\varphi + A^2\sin^2\sqrt{2}\varphi},$$
$$w_{ij} = w_{ij}^{(t)} e_t + \sum_{s\neq t} w_{ij}^{(s)} e_s = \frac{\sqrt{2}(1-A^2)\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi + A^2\sin^2\sqrt{2}\varphi} + \sum_{s\neq t} z_{ij}^{(s)} \frac{\cos^2\sqrt{2}c' + A^2\sin^2\sqrt{2}c'}{\cos^2\sqrt{2}\varphi + A^2\sin^2\sqrt{2}\varphi}$$

$$w_{pi}^{(u)} = z_{pi}^{(u)} \frac{\cos^2\sqrt{2}c' + A^2\sin^2\sqrt{2}c'}{\sin\sqrt{2}(c' + d_{pi}^{(u)\prime})} \frac{\sin\sqrt{2}(\varphi + d_{pi}^{(u)\prime})}{\cos^2\sqrt{2}\varphi + A^2\sin^2\sqrt{2}\varphi}, \quad u = 1, \cdots, n_{pi}, \quad p < i,$$

$$w_{pj}^{(u)} = z_{pj}^{(u)} \frac{\cos^2\sqrt{2}c' + A^2\sin^2\sqrt{2}c'}{\sin\sqrt{2}(c' + d_{pj}^{(u)\prime})} \frac{\sin\sqrt{2}(\varphi + d_{pj}^{(u)\prime})}{\cos^2\sqrt{2}\varphi + A^2\sin^2\sqrt{2}\varphi}, \quad u = 1, \cdots, n_{pj}, \quad p < i,$$

$$w_{ip}^{(u)} = z_{ip}^{(u)} \frac{\cos^2\sqrt{2}c' + A^2\sin^2\sqrt{2}c'}{\sin\sqrt{2}(c' + d_{ip}^{(u)\prime})} \frac{\sin\sqrt{2}(\varphi + d_{ip}^{(u)\prime})}{\cos^2\sqrt{2}\varphi + A^2\sin^2\sqrt{2}\varphi}, \quad u = 1, \cdots, n_{ip}, \quad i < p < j,$$

$$w_{pj}^{(u)} = z_{pj}^{(u)} \frac{\cos^2\sqrt{2}c' + A^2\sin^2\sqrt{2}c'}{\sin\sqrt{2}(c' + d_{pj}^{(u)\prime})} \frac{\sin\sqrt{2}(\varphi + d_{pj}^{(u)\prime})}{\cos^2\sqrt{2}\varphi + A^2\sin^2\sqrt{2}\varphi}, \quad u = 1, \cdots, n_{pj}, \quad i < p < j,$$

$$w_{ip}^{(u)} = z_{ip}^{(u)} \frac{\cos^2\sqrt{2}c' + A^2\sin^2\sqrt{2}c'}{\sin\sqrt{2}(c' + d_{ip}^{(u)\prime})} \frac{\sin\sqrt{2}(\varphi + d_{ip}^{(u)\prime})}{\cos^2\sqrt{2}\varphi + A^2\sin^2\sqrt{2}\varphi}, \quad u = 1, \cdots, n_{ip}, \quad p > j,$$

$$w_{jp}^{(u)} = z_{jp}^{(u)} \frac{\cos^2\sqrt{2}c' + A^2\sin^2\sqrt{2}c'}{\sin\sqrt{2}(c' + d_{jp}^{(u)\prime})} \frac{\sin\sqrt{2}(\varphi + d_{jp}^{(u)\prime})}{\cos^2\sqrt{2}\varphi + A^2\sin^2\sqrt{2}\varphi}, \quad u = 1, \cdots, n_{jp}, \quad p > j,$$

$$v_i^{(u)} = d_i^{(u)} \frac{\cos^2\sqrt{2}c' + A^2\sin^2\sqrt{2}c'}{\sin\sqrt{2}(c' + d_i^{(u)\prime})} \frac{\sin\sqrt{2}(\varphi + d_i^{(u)\prime})}{\cos^2\sqrt{2}\varphi + A^2\sin^2\sqrt{2}\varphi}, \quad u = 1, \cdots, m_i,$$

$$v_j^{(u)} = d_j^{(u)} \frac{\cos^2\sqrt{2}c' + A^2\sin^2\sqrt{2}c'}{\sin\sqrt{2}(c' + d_j^{(u)\prime})} \frac{\sin\sqrt{2}(\varphi + d_j^{(u)\prime})}{\cos^2\sqrt{2}\varphi + A^2\sin^2\sqrt{2}\varphi}, \quad u = 1, \cdots, m_j,$$

$$r_p = g_0(\varphi) + a_0, \quad p < i,$$
$$r_p = g_1(\varphi) + a_1, \quad i < p < j,$$
$$r_p = g_2(\varphi) + a_2, \quad p > j,$$
$$w_{pk} = g_3(\varphi) + a_3, \quad p < k < i,$$
$$w_{pk} = g_4(\varphi) + a_4, \quad p < i < k < j,$$
$$w_{pk} = g_5(\varphi) + a_5, \quad p < i < j < k,$$
$$w_{pk} = g_6(\varphi) + a_6, \quad i < p < k < j,$$
$$w_{pk} = g_7(\varphi) + a_7, \quad i < p < j < k,$$

$$
\begin{aligned}
w_{pk} &= g_8(\varphi) + a_8, \quad j < p < k, \\
v_p &= g_9(\varphi) + a_9, \quad p < i, \\
v_p &= g_{10}(\varphi) + a_{10}, \quad i < p < j, \\
v_p &= g_{11}(\varphi) + a_{11}, \quad p > j.
\end{aligned}
$$

其中

$$
c = \frac{\sum\limits_{s \neq t} {z_{ij}^{(s)}}^2 - s_i s_j}{(s_i s_j - z_{ij} {z_{ij}}' - 2)^2},
$$

$$
c' = \frac{1}{\sqrt{2}} \operatorname{arctg} \sqrt{2} \frac{\sqrt{1-8c} + \sqrt{1 - 8c - 4c {z_{ij}^{(t)}}^2}}{1 - \sqrt{1-8c} z_{ij}^{(t)}}, \quad A = \frac{1 - \sqrt{1-8c}}{2\sqrt{2c}},
$$

$$
\varphi = \theta + c',
$$

在 $w_{pi}^{(u)} (p < i)$ 的表达式中,

$$
{d_{pi}^{(u)}}' = \operatorname{arcctg} \left(\frac{[s_i \sum\limits_s z_{pj} A_{pi}^{sj} e_t' e_s + \sum\limits_{s,r} z_{pi}^{(s)} z_{ij} (A_{pi}^{rj})' A_{pi}^{sj} e_t' e_r] e_u'}{\sqrt{2} z_{pi}^{(u)}} - \frac{(1-A^2) \sin 2\sqrt{2}c'}{\cos^2 \sqrt{2}c' + A^2 \sin^2 \sqrt{2}c'} \right) - c';
$$

在 $w_{pj}^{(u)} (p < i)$ 的表达式中,

$$
{d_{pj}^{(u)}}' = \operatorname{arcctg} \left(\frac{[s_j \sum\limits_s z_{pi}^{(s)} e_t (A_{pi}^{sj})' + \sum\limits_s z_{pj} A_{pi}^{sj} e_t' z_{ij} (A_{pi}^{sj})'] e_u'}{\sqrt{2} z_{pj}^{(u)}} - \frac{(1-A^2) \sin 2\sqrt{2}c'}{\cos^2 \sqrt{2}c' + A^2 \sin^2 \sqrt{2}c'} \right) - c';
$$

在 $w_{ip}^{(u)} (i < p < j)$ 的表达式中,

$$
{d_{ip}^{(u)}}' = \operatorname{arcctg} \left(\frac{[s_i \sum\limits_s e_t A_{ip}^{sj} z_{pj}' e_s + \sum\limits_{s,r} z_{ip}^{(s)} z_{ij} A_{ip}^{rj} (A_{ip}^{sj})' e_t' e_r] e_u'}{\sqrt{2} z_{ip}^{(u)}} - \frac{(1-A^2) \sin 2\sqrt{2}c'}{\cos^2 \sqrt{2}c' + A^2 \sin^2 \sqrt{2}c'} \right) - c';
$$

在 $w_{pj}^{(u)} (i < p < j)$ 的表达式中,

$$
{d_{pj}^{(u)}}' = \operatorname{arcctg} \left(\frac{[s_j \sum\limits_s z_{ip}^{(s)} e_t A_{ip}^{sj} + \sum\limits_s e_t A_{ip}^{sj} z_{pj}' z_{ij} A_{ip}^{sj}] e_u'}{\sqrt{2} z_{pj}^{(u)}} - \frac{(1-A^2) \sin 2\sqrt{2}c'}{\cos^2 \sqrt{2}c' + A^2 \sin^2 \sqrt{2}c'} \right) - c';
$$

在 $w_{ip}^{(u)} (p > j)$ 的表达式中,

$$
{d_{ip}^{(u)}}' = \operatorname{arcctg} \left(\frac{[s_i z_{jp} (A_{ij}^{tp})' + \sum\limits_s z_{ij}^{(s)} z_{ip} A_{ij}^{tp} (A_{ij}^{sp})'] e_u'}{\sqrt{2} z_{ip}^{(u)}} - \frac{(1-A^2) \sin 2\sqrt{2}c'}{\cos^2 \sqrt{2}c' + A^2 \sin^2 \sqrt{2}c'} \right) - c';
$$

在 $w_{jp}^{(u)} (p > j)$ 的表达式中,

$$
{d_{jp}^{(u)}}' = \operatorname{arcctg} \left(\frac{[s_j z_{ip} A_{ij}^{tp} + \sum\limits_s z_{ij}^{(s)} z_{jp} (A_{ij}^{tp})' A_{ij}^{sp}] e_u'}{\sqrt{2} z_{jp}^{(u)}} - \frac{(1-A^2) \sin 2\sqrt{2}c'}{\cos^2 \sqrt{2}c' + A^2 \sin^2 \sqrt{2}c'} \right) - c';
$$

在 $v_i^{(u)}$ 的表达式中，

$$d_i^{(u)'} = \operatorname{arcctg}\left(\frac{[s_i u_j \overline{(Q_{ij}^{(t)})}' + \sum_s z_{ij}^{(s)} u_i Q_{ij}^{(t)} \overline{(Q_{ij}^{(s)})}']e_u'}{\sqrt{2}u_i^{(u)}} - \frac{(1-A^2)\sin 2\sqrt{2}c'}{\cos^2\sqrt{2}c' + A^2\sin^2\sqrt{2}c'}\right) - c';$$

在 $v_j^{(u)}$ 的表达式中，

$$d_j^{(u)'} = \operatorname{arcctg}\left(\frac{[s_j u_i Q_{ij}^{(t)} + \sum_s z_{ij}^{(s)} u_j \overline{(Q_{ij}^{(t)})}' Q_{ij}^{(s)}]e_u'}{\sqrt{2}u_j^{(u)}} - \frac{(1-A^2)\sin 2\sqrt{2}c'}{\cos^2\sqrt{2}c' + A^2\sin^2\sqrt{2}c'}\right) - c';$$

在 $r_p(p < i)$ 的表达式中，

$$\begin{aligned} g_0(\varphi) = \frac{1}{\sqrt{2}}\sum_{s,r}[&\frac{\cos\sqrt{2}d_{pi}^{(s)'}\cos\sqrt{2}d_{pj}^{(r)'}}{A^3}(\operatorname{arctg} A\operatorname{tg}\sqrt{2}\varphi - \frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi + A^2\sin^2\sqrt{2}\varphi}) \\ &+ \frac{\sin\sqrt{2}(d_{pi}^{(s)'} + d_{pj}^{(r)'})}{1-A^2}\frac{1}{\cos^2\sqrt{2}\varphi + A^2\sin^2\sqrt{2}\varphi} + \frac{\sin\sqrt{2}d_{pi}^{(s)'}\sin\sqrt{2}d_{pj}^{(r)'}}{A}(\operatorname{arctg} A\operatorname{tg}\sqrt{2}\varphi \\ &+ \frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi + A^2\sin^2\sqrt{2}\varphi})]z_{pi}^{(s)}z_{pj}^{(r)}\frac{(\cos^2\sqrt{2}c' + A^2\sin^2\sqrt{2}c')^2}{\sin\sqrt{2}(c' + d_{pi}^{(s)'})\sin\sqrt{2}(c' + d_{pj}^{(r)'})}e_r A_{pi}^{sj} e_t', \end{aligned}$$

$a_0 = s_p - g_0(c');$

在 $r_p(i < p < j)$ 的表达式中，

$$\begin{aligned} g_1(\varphi) = \frac{1}{\sqrt{2}}\sum_{s,r}[&\frac{\cos\sqrt{2}d_{ip}^{(s)'}\cos\sqrt{2}d_{pj}^{(r)'}}{A^3}(\operatorname{arctg} A\operatorname{tg}\sqrt{2}\varphi - \frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi + A^2\sin^2\sqrt{2}\varphi}) \\ &+ \frac{\sin\sqrt{2}(d_{ip}^{(s)'} + d_{pj}^{(r)'})}{1-A^2}\frac{1}{\cos^2\sqrt{2}\varphi + A^2\sin^2\sqrt{2}\varphi} + \frac{\sin\sqrt{2}d_{ip}^{(s)'}\sin\sqrt{2}d_{pj}^{(r)'}}{A}(\operatorname{arctg} A\operatorname{tg}\sqrt{2}\varphi \\ &+ \frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi + A^2\sin^2\sqrt{2}\varphi})]z_{ip}^{(s)}z_{pj}^{(r)}\frac{(\cos^2\sqrt{2}c' + A^2\sin^2\sqrt{2}c')^2}{\sin\sqrt{2}(c' + d_{ip}^{(s)'})\sin\sqrt{2}(c' + d_{pj}^{(r)'})}e_r (A_{ip}^{sj})' e_t', \end{aligned}$$

$a_1 = s_p - g_1(c');$

在 $r_p(p > j)$ 的表达式中，

$$\begin{aligned} g_2(\varphi) = \frac{1}{\sqrt{2}}\sum_{s,r}[&\frac{\cos\sqrt{2}d_{ip}^{(s)'}\cos\sqrt{2}d_{jp}^{(r)'}}{A^3}(\operatorname{arctg} A\operatorname{tg}\sqrt{2}\varphi - \frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi + A^2\sin^2\sqrt{2}\varphi}) \\ &+ \frac{\sin\sqrt{2}(d_{ip}^{(s)'} + d_{jp}^{(r)'})}{1-A^2}\frac{1}{\cos^2\sqrt{2}\varphi + A^2\sin^2\sqrt{2}\varphi} + \frac{\sin\sqrt{2}d_{ip}^{(s)'}\sin\sqrt{2}d_{jp}^{(r)'}}{A}(\operatorname{arctg} A\operatorname{tg}\sqrt{2}\varphi \\ &+ \frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi + A^2\sin^2\sqrt{2}\varphi})]z_{ip}^{(s)}z_{jp}^{(r)}\frac{(\cos^2\sqrt{2}c' + A^2\sin^2\sqrt{2}c')^2}{\sin\sqrt{2}(c' + d_{ip}^{(s)'})\sin\sqrt{2}(c' + d_{jp}^{(r)'})}e_s A_{ij}^{tp} e_r', \end{aligned}$$

$a_2 = s_p - g_2(c');$

在 $w_{pk}(p<k<i)$ 的表达式中，

$$g_3(\varphi)=\frac{1}{2\sqrt{2}}\sum_{s,r,u,v}[\frac{\cos\sqrt{2}d_{pj}^{(u)'}\cos\sqrt{2}d_{ki}^{(v)'}}{A^3}(\mathrm{arctg}\,A\mathrm{tg}\,\sqrt{2}\varphi-\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})$$

$$+\frac{\sin\sqrt{2}(d_{pj}^{(u)'}+d_{ki}^{(v)'})}{1-A^2}\frac{1}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi}$$

$$+\frac{\sin\sqrt{2}d_{pj}^{(u)'}\sin\sqrt{2}d_{ki}^{(v)'}}{A}(\mathrm{arctg}\,A\mathrm{tg}\,\sqrt{2}\varphi+\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})]$$

$$z_{pj}^{(u)}z_{ki}^{(v)}\frac{(\cos^2\sqrt{2}c'+A^2\sin^2\sqrt{2}c')^2}{\sin\sqrt{2}(c'+d_{pj}^{(u)'})\sin\sqrt{2}(c'+d_{ki}^{(v)'})}e_uA_{pi}^{sj}e_t'e_sA_{pk}^{ri}e_v'e_r$$

$$+\frac{1}{2\sqrt{2}}\sum_{s,r,u,v}[\frac{\cos\sqrt{2}d_{kj}^{(u)'}\cos\sqrt{2}d_{pi}^{(v)'}}{A^3}(\mathrm{arctg}\,A\mathrm{tg}\,\sqrt{2}\varphi-\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})$$

$$+\frac{\sin\sqrt{2}(d_{kj}^{(u)'}+d_{pi}^{(v)'})}{1-A^2}\frac{1}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi}$$

$$+\frac{\sin\sqrt{2}d_{kj}^{(u)'}\sin\sqrt{2}d_{pi}^{(v)'}}{A}(\mathrm{arctg}\,A\mathrm{tg}\,\sqrt{2}\varphi+\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})]$$

$$z_{kj}^{(u)}z_{pi}^{(v)}\frac{(\cos^2\sqrt{2}c'+A^2\sin^2\sqrt{2}c')^2}{\sin\sqrt{2}(c'+d_{kj}^{(u)'})\sin\sqrt{2}(c'+d_{pi}^{(v)'})}e_uA_{ki}^{sj}e_t'e_s(A_{pk}^{ri})'e_v'e_r,$$

$a_3=z_{pk}-g_3(c')$;

在 $w_{pk}(p<i<k<j)$ 的表达式中，

$$g_4(\varphi)=\frac{1}{2\sqrt{2}}\sum_{s,r,u}[\frac{\cos\sqrt{2}d_{pi}^{(r)'}\cos\sqrt{2}d_{kj}^{(u)'}}{A^3}(\mathrm{arctg}\,A\mathrm{tg}\,\sqrt{2}\varphi-\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})$$

$$+\frac{\sin\sqrt{2}(d_{pi}^{(r)'}+d_{kj}^{(u)'})}{1-A^2}\frac{1}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi}+\frac{\sin\sqrt{2}d_{pi}^{(r)'}\sin\sqrt{2}d_{kj}^{(u)'}}{A}(\mathrm{arctg}\,A\mathrm{tg}\,\sqrt{2}\varphi$$

$$+\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})]z_{pi}^{(r)}z_{kj}^{(u)}\frac{(\cos^2\sqrt{2}c'+A^2\sin^2\sqrt{2}c')^2}{\sin\sqrt{2}(c'+d_{pi}^{(r)'})\sin\sqrt{2}(c'+d_{kj}^{(u)'})}e_tA_{ik}^{sj}e_u'e_s(A_{pi}^{rk})'$$

$$+\frac{1}{2\sqrt{2}}\sum_{s,r,u}[\frac{\cos\sqrt{2}d_{pj}^{(r)'}\cos\sqrt{2}d_{ik}^{(u)'}}{A^3}(\mathrm{arctg}\,A\mathrm{tg}\,\sqrt{2}\varphi-\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})$$

$$+\frac{\sin\sqrt{2}(d_{pj}^{(r)'}+d_{ik}^{(u)'})}{1-A^2}\frac{1}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi}+\frac{\sin\sqrt{2}d_{pj}^{(r)'}\sin\sqrt{2}d_{ik}^{(u)'}}{A}(\mathrm{arctg}\,A\mathrm{tg}\,\sqrt{2}\varphi$$

$$+\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})]z_{pj}^{(r)}z_{ik}^{(u)}\frac{(\cos^2\sqrt{2}c'+A^2\sin^2\sqrt{2}c')^2}{\sin\sqrt{2}(c'+d_{pj}^{(r)'})\sin\sqrt{2}(c'+d_{ik}^{(u)'})}e_rA_{pi}^{sj}e_t'e_u(A_{pi}^{sk})',$$

$a_4=z_{pk}-g_4(c')$;

在 $w_{pk}(p<i<j<k)$ 的表达式中，

$$g_5(\varphi)=\frac{1}{2\sqrt{2}}\sum_{s,r}[\frac{\cos\sqrt{2}d_{pi}^{(s)'}\cos\sqrt{2}d_{jk}^{(r)'}}{A^3}(\text{arctg}\,A\text{tg}\sqrt{2}\varphi-\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})$$

$$+\frac{\sin\sqrt{2}(d_{pi}^{(s)'}+d_{jk}^{(r)'})}{1-A^2}\frac{1}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi}+\frac{\sin\sqrt{2}d_{pi}^{(s)'}\sin\sqrt{2}d_{jk}^{(r)'}}{A}(\text{arctg}\,A\text{tg}\sqrt{2}\varphi$$

$$+\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})]z_{pi}^{(s)}z_{jk}^{(r)}\frac{(\cos^2\sqrt{2}c'+A^2\sin^2\sqrt{2}c')^2}{\sin\sqrt{2}(c'+d_{pi}^{(s)'})\sin\sqrt{2}(c'+d_{jk}^{(r)'})}e_r(A_{ij}^{tk})'(A_{pi}^{sk})'$$

$$+\frac{1}{2\sqrt{2}}\sum_{s,r,u}[\frac{\cos\sqrt{2}d_{pj}^{(r)'}\cos\sqrt{2}d_{ik}^{(u)'}}{A^3}(\text{arctg}\,A\text{tg}\sqrt{2}\varphi-\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})$$

$$+\frac{\sin\sqrt{2}(d_{pj}^{(r)'}+d_{ik}^{(u)'})}{1-A^2}\frac{1}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi}+\frac{\sin\sqrt{2}d_{pj}^{(r)'}\sin\sqrt{2}d_{ik}^{(u)'}}{A}(\text{arctg}\,A\text{tg}\sqrt{2}\varphi$$

$$+\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})]z_{pj}^{(r)}z_{ik}^{(u)}\frac{(\cos^2\sqrt{2}c'+A^2\sin^2\sqrt{2}c')^2}{\sin\sqrt{2}(c'+d_{pj}^{(r)'})\sin\sqrt{2}(c'+d_{ik}^{(u)'})}e_rA_{pi}^{sj}e_t'e_u(A_{pi}^{sk})',$$

$a_5=z_{pk}-g_5(c')$;

在 $w_{pk}(i<p<k<j)$ 的表达式中，

$$g_6(\varphi)=\frac{1}{2\sqrt{2}}\sum_{s,r,u}[\frac{\cos\sqrt{2}d_{ip}^{(r)'}\cos\sqrt{2}d_{kj}^{(u)'}}{A^3}(\text{arctg}\,A\text{tg}\sqrt{2}\varphi-\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})$$

$$+\frac{\sin\sqrt{2}(d_{ip}^{(r)'}+d_{kj}^{(u)'})}{1-A^2}\frac{1}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi}+\frac{\sin\sqrt{2}d_{ip}^{(r)'}\sin\sqrt{2}d_{kj}^{(u)'}}{A}(\text{arctg}\,A\text{tg}\sqrt{2}\varphi$$

$$+\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})]z_{ip}^{(r)}z_{kj}^{(u)}\frac{(\cos^2\sqrt{2}c'+A^2\sin^2\sqrt{2}c')^2}{\sin\sqrt{2}(c'+d_{ip}^{(r)'})\sin\sqrt{2}(c'+d_{kj}^{(u)'})}e_tA_{ik}^{sj}e_u'e_sA_{ip}^{rk}$$

$$+\frac{1}{2\sqrt{2}}\sum_{s,r,u}[\frac{\cos\sqrt{2}d_{pj}^{(r)'}\cos\sqrt{2}d_{ik}^{(u)'}}{A^3}(\text{arctg}\,A\text{tg}\sqrt{2}\varphi-\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})$$

$$+\frac{\sin\sqrt{2}(d_{pj}^{(r)'}+d_{ik}^{(u)'})}{1-A^2}\frac{1}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi}+\frac{\sin\sqrt{2}d_{pj}^{(r)'}\sin\sqrt{2}d_{ik}^{(u)'}}{A}(\text{arctg}\,A\text{tg}\sqrt{2}\varphi$$

$$+\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})]z_{pj}^{(r)}z_{ik}^{(u)}\frac{(\cos^2\sqrt{2}c'+A^2\sin^2\sqrt{2}c')^2}{\sin\sqrt{2}(c'+d_{pj}^{(r)'})\sin\sqrt{2}(c'+d_{ik}^{(u)'})}e_tA_{ip}^{sj}e_r'e_uA_{ip}^{sk},$$

$a_6=z_{pk}-g_6(c')$;

在 $w_{pk}(i<p<j<k)$ 的表达式中，

$$g_7(\varphi)=\frac{1}{2\sqrt{2}}\sum_{s,r}[\frac{\cos\sqrt{2}d_{ip}^{(s)'}\cos\sqrt{2}d_{jk}^{(r)'}}{A^3}(\mathrm{arctg}\,A\mathrm{tg}\sqrt{2}\varphi-\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})$$
$$+\frac{\sin\sqrt{2}(d_{ip}^{(s)'}+d_{jk}^{(r)'})}{1-A^2}\frac{1}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi}+\frac{\sin\sqrt{2}d_{ip}^{(s)'}\sin\sqrt{2}d_{jk}^{(r)'}}{A}(\mathrm{arctg}\,A\mathrm{tg}\sqrt{2}\varphi$$
$$+\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})]z_{ip}^{(s)}z_{jk}^{(r)}\frac{(\cos^2\sqrt{2}c'+A^2\sin^2\sqrt{2}c')^2}{\sin\sqrt{2}(c'+d_{ip}^{(s)'})\sin\sqrt{2}(c'+d_{jk}^{(r)'})}e_r(A_{ij}^{tk})'A_{ip}^{sk}$$
$$+\frac{1}{2\sqrt{2}}\sum_{s,r,u}[\frac{\cos\sqrt{2}d_{pj}^{(r)'}\cos\sqrt{2}d_{ik}^{(u)'}}{A^3}(\mathrm{arctg}\,A\mathrm{tg}\sqrt{2}\varphi-\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})$$
$$+\frac{\sin\sqrt{2}(d_{pj}^{(r)'}+d_{ik}^{(u)'})}{1-A^2}\frac{1}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi}+\frac{\sin\sqrt{2}d_{pj}^{(r)'}\sin\sqrt{2}d_{ik}^{(u)'}}{A}(\mathrm{arctg}\,A\mathrm{tg}\sqrt{2}\varphi$$
$$+\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})]z_{pj}^{(r)}z_{ik}^{(u)}\frac{(\cos^2\sqrt{2}c'+A^2\sin^2\sqrt{2}c')^2}{\sin\sqrt{2}(c'+d_{pj}^{(r)'})\sin\sqrt{2}(c'+d_{ik}^{(u)'})}e_tA_{ip}^{sj}c_r'e_uA_{ip}^{sk},$$

$a_7=z_{pk}-g_7(c')$;

在 $w_{pk}(j<p<k)$ 的表达式中，

$$g_8(\varphi)=\frac{1}{2\sqrt{2}}\sum_{s,r,u}[\frac{\cos\sqrt{2}d_{jp}^{(r)'}\cos\sqrt{2}d_{ik}^{(u)'}}{A^3}(\mathrm{arctg}\,A\mathrm{tg}\sqrt{2}\varphi-\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})$$
$$+\frac{\sin\sqrt{2}(d_{jp}^{(r)'}+d_{ik}^{(u)'})}{1-A^2}\frac{1}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi}+\frac{\sin\sqrt{2}d_{jp}^{(r)'}\sin\sqrt{2}d_{ik}^{(u)'}}{A}(\mathrm{arctg}\,A\mathrm{tg}\sqrt{2}\varphi$$
$$+\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})]z_{jp}^{(r)}z_{ik}^{(u)}\frac{(\cos^2\sqrt{2}c'+A^2\sin^2\sqrt{2}c')^2}{\sin\sqrt{2}(c'+d_{jp}^{(r)'})\sin\sqrt{2}(c'+d_{ik}^{(u)'})}e_r(A_{ij}^{tp})'e_s'e_uA_{ip}^{sk}$$
$$+\frac{1}{2\sqrt{2}}\sum_{s,r}[\frac{\cos\sqrt{2}d_{ip}^{(s)'}\cos\sqrt{2}d_{jk}^{(r)'}}{A^3}(\mathrm{arctg}\,A\mathrm{tg}\sqrt{2}\varphi-\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})$$
$$+\frac{\sin\sqrt{2}(d_{ip}^{(s)'}+d_{jk}^{(r)'})}{1-A^2}\frac{1}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi}+\frac{\sin\sqrt{2}d_{ip}^{(s)'}\sin\sqrt{2}d_{jk}^{(r)'}}{A}(\mathrm{arctg}\,A\mathrm{tg}\sqrt{2}\varphi$$
$$+\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})]z_{ip}^{(s)}z_{jk}^{(r)}\frac{(\cos^2\sqrt{2}c'+A^2\sin^2\sqrt{2}c')^2}{\sin\sqrt{2}(c'+d_{ip}^{(s)'})\sin\sqrt{2}(c'+d_{jk}^{(r)'})}e_r(A_{ij}^{tk})'A_{ip}^{sk},$$

$a_8=z_{pk}-g_8(c')$;

在 $v_p(p<i)$ 的表达式中,

$$g_9(\varphi)=\frac{1}{2\sqrt{2}}\sum_{s,r,u}[\frac{\cos\sqrt{2}d_{pj}^{(r)'}\cos\sqrt{2}d_i^{(u)'}}{A^3}(\operatorname{arctg} A\operatorname{tg}\sqrt{2}\varphi-\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})$$
$$+\frac{\sin\sqrt{2}(d_{pj}^{(r)'}+d_i^{(u)'})}{1-A^2}\frac{1}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi}+\frac{\sin\sqrt{2}d_{pj}^{(r)'}\sin\sqrt{2}d_i^{(u)'}}{A}(\operatorname{arctg} A\operatorname{tg}\sqrt{2}\varphi$$
$$+\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})]z_{pj}^{(r)}u_i^{(u)}\frac{(\cos^2\sqrt{2}c'+A^2\sin^2\sqrt{2}c')^2}{\sin\sqrt{2}(c'+d_{pj}^{(r)'})\sin\sqrt{2}(c'+d_i^{(u)'})}e_rA_{pi}^{sj}e_t'e_u\overline{(Q_{pi}^{(s)})'}$$
$$+\frac{1}{2\sqrt{2}}\sum_{s,r}[\frac{\cos\sqrt{2}d_{pi}^{(s)'}\cos\sqrt{2}d_j^{(r)'}}{A^3}(\operatorname{arctg} A\operatorname{tg}\sqrt{2}\varphi-\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})$$
$$+\frac{\sin\sqrt{2}(d_{pi}^{(s)'}+d_j^{(r)'})}{1-A^2}\frac{1}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi}+\frac{\sin\sqrt{2}d_{pi}^{(s)'}\sin\sqrt{2}d_j^{(r)'}}{A}(\operatorname{arctg} A\operatorname{tg}\sqrt{2}\varphi$$
$$+\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})]z_{pi}^{(s)}u_j^{(r)}\frac{(\cos^2\sqrt{2}c'+A^2\sin^2\sqrt{2}c')^2}{\sin\sqrt{2}(c'+d_{pi}^{(s)'})\sin\sqrt{2}(c'+d_j^{(r)'})}e_r\overline{(Q_{ij}^{(t)})'}\,\overline{(Q_{pi}^{(s)})'},$$

$a_9=u_p-g_9(c')$;

在 $v_p(i<p<j)$ 的表达式中,

$$g_{10}(\varphi)=\frac{1}{2\sqrt{2}}\sum_{s,r,u}[\frac{\cos\sqrt{2}d_{pj}^{(r)'}\cos\sqrt{2}d_i^{(u)'}}{A^3}(\operatorname{arctg} A\operatorname{tg}\sqrt{2}\varphi-\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})$$
$$+\frac{\sin\sqrt{2}(d_{pj}^{(r)'}+d_i^{(u)'})}{1-A^2}\frac{1}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi}+\frac{\sin\sqrt{2}d_{pj}^{(r)'}\sin\sqrt{2}d_i^{(u)'}}{A}(\operatorname{arctg} A\operatorname{tg}\sqrt{2}\varphi$$
$$+\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})]z_{pj}^{(r)}u_i^{(u)}\frac{(\cos^2\sqrt{2}c'+A^2\sin^2\sqrt{2}c')^2}{\sin\sqrt{2}(c'+d_{pj}^{(r)'})\sin\sqrt{2}(c'+d_i^{(u)'})}e_tA_{ip}^{sj}e_r'e_uQ_{ip}^{(s)}$$
$$+\frac{1}{2\sqrt{2}}\sum_{s,r}[\frac{\cos\sqrt{2}d_{ip}^{(s)'}\cos\sqrt{2}d_j^{(r)'}}{A^3}(\operatorname{arctg} A\operatorname{tg}\sqrt{2}\varphi-\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})$$
$$+\frac{\sin\sqrt{2}(d_{ip}^{(s)'}+d_j^{(r)'})}{1-A^2}\frac{1}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi}+\frac{\sin\sqrt{2}d_{ip}^{(s)'}\sin\sqrt{2}d_j^{(r)'}}{A}(\operatorname{arctg} A\operatorname{tg}\sqrt{2}\varphi$$
$$+\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})]z_{ip}^{(s)}u_j^{(r)}\frac{(\cos^2\sqrt{2}c'+A^2\sin^2\sqrt{2}c')^2}{\sin\sqrt{2}(c'+d_{ip}^{(s)'})\sin\sqrt{2}(c'+d_j^{(r)'})}e_r\overline{(Q_{ij}^{(t)})'}Q_{ip}^{(s)},$$

$a_{10}=u_p-g_{10}(c')$;

在 $v_p(p>j)$ 的表达式中,

$$g_{11}(\varphi)=\frac{1}{2\sqrt{2}}\sum_{s,r,u}[\frac{\cos\sqrt{2}d_{jp}^{(r)'}\cos\sqrt{2}d_i^{(u)'}}{A^3}(\text{arctg}\,A\text{tg}\,\sqrt{2}\varphi-\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})$$

$$+\frac{\sin\sqrt{2}(d_{jp}^{(r)'}+d_i^{(u)'})}{1-A^2}\frac{1}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi}+\frac{\sin\sqrt{2}d_{jp}^{(r)'}\sin\sqrt{2}d_i^{(u)'}}{A}(\text{arctg}\,A\text{tg}\,\sqrt{2}\varphi$$

$$+\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})]z_{jp}^{(r)}u_i^{(u)}\frac{(\cos^2\sqrt{2}c'+A^2\sin^2\sqrt{2}c')^2}{\sin\sqrt{2}(c'+d_{jp}^{(r)'})\sin\sqrt{2}(c'+d_i^{(u)'})}e_r(A_{ij}^{tp})'e_s'e_uQ_{ip}^{(s)}$$

$$+\frac{1}{2\sqrt{2}}\sum_{s,r}[\frac{\cos\sqrt{2}d_{ip}^{(s)'}\cos\sqrt{2}d_j^{(r)'}}{A^3}(\text{arctg}\,A\text{tg}\,\sqrt{2}\varphi-\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})$$

$$+\frac{\sin\sqrt{2}(d_{ip}^{(s)'}+d_j^{(r)'})}{1-A^2}\frac{1}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi}+\frac{\sin\sqrt{2}d_{ip}^{(s)'}\sin\sqrt{2}d_j^{(r)'}}{A}(\text{arctg}\,A\text{tg}\,\sqrt{2}\varphi$$

$$+\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})]z_{ip}^{(s)}u_j^{(r)}\frac{(\cos^2\sqrt{2}c'+A^2\sin^2\sqrt{2}c')^2}{\sin\sqrt{2}(c'+d_{ip}^{(s)'})\sin\sqrt{2}(c'+d_j^{(r)'})}e_r\overline{(Q_{ij}^{(t)})'}Q_{ip}^{(s)},$$

$a_{11}=u_p-g_{11}(c')$.

第二章　记号及预备知识

首先给出正规 Siegel 的定义.

给定非负整数 n_{ij}, $1 \le i \le j \le N$, 其中 $n_{ii} = 1$, $1 \le i \le N$. 记 $n = \sum\limits_{1\le i\le j\le N} n_{ij}$. $\mathbb{C}^n$ 中点 z 的坐标排为

$$z = (s_1, z_2, s_2, \cdots, z_N, s_N), \quad z_j = (z_{1j}, z_{2j}, \cdots, z_{j-1,j}),$$

其中

$$s_i \in \mathbb{C}, \quad z_{ij} \in \mathbb{C}^{n_{ij}}, \quad z_{ij} = (z_{ij}^{(1)}, \cdots, z_{ij}^{(n_{ij})}).$$

当 z 的坐标中 $s_j = 1$, 其余坐标全为零时，记 z 为 e_{jj}; 当 z 的坐标中 $z_{ij}^{(t)} = 1$, 其余坐标全为零时，记 z 为 $e_{ij}^{(t)}$;

给定非负整数 m_i, $1 \le i \le N$, 记 $m = \sum\limits_{i=1}^{N} m_i$. $\mathbb{C}^m$ 中点 u 的坐标排为

$$u = (u_1, \cdots, u_N),$$

其中

$$u_j \in \mathbb{C}^{m_j}, \quad u_j = (u_j^{(1)}, \cdots, u_j^{(m_j)}), \quad 1 \le j \le N$$

当 u 的坐标中 $u_j^{(t)} = 1$, 其余坐标全为零时，记 u 为 $e_j^{(t)}$;

记 $e_i = (0, \cdots, 1, \cdots, 0)$, 其中 1 为第 i 个分量， $n \times m$ 阶矩阵 A 的第 i 行第 j 列元素可记为 $e_i A e_j'$, 记 E_{sr} 为第 s 行， r 列元素为 1, 其余元素为 0 的矩阵.

定义 2.1　设 A_{ij}^{tk}, $t = 1, \cdots, n_{ij}$ 为 $n_{ik} \times n_{jk}$ 阶实矩阵， $1 \le i < j \le N$; $Q_{ij}^{(t)}$, $t = 1, \cdots, n_{ij}$ 为 $m_i \times m_j$ 阶复矩阵， $1 \le i < j \le N$. 矩阵组 $\{A_{ij}^{tk}, Q_{ij}^{(t)}, 1 \le t \le n_{ij}, 1 \le i < j < k \le N\}$ 称为 N 矩阵组. 如果它们适合以下 8 个条件

(1)　$n_{ik} = 0$ 蕴含 $n_{ij} n_{jk} = 0$, $j = i+1, \cdots, k-1$;

(2)

$$(A_{ij}^{sk})' A_{ij}^{tk} + (A_{ij}^{tk})' A_{ij}^{sk} = 2\delta_{s,t} I^{(n_{jk})},$$

其中 $1 \le s, t \le n_{ij}$, $1 \le i < j < k \le N$;

(3)

$$A_{ij}^{sk} A_{jp}^{tk} = \sum_r (e_r A_{ij}^{sp} e_t') A_{ip}^{rk},$$

其中 $1 \le s \le n_{ij}$, $1 \le t \le n_{jp}$, $1 \le i < j < p < k \le N$;

(4)

$$(A_{ij}^{sk})' A_{ip}^{tk} = \sum_r (e_t A_{ij}^{sp} e_r') A_{jp}^{rk},$$

其中 $1 \le s \le n_{ij}$, $1 \le t \le n_{ip}$, $1 \le i < j < p < k \le N$;

(5)　$m_k = 0$ 蕴含 $n_{ij} m_j = 0$, $j = i+1, \cdots, N$;

(6)

$$\overline{(Q_{ij}^{(s)})}' Q_{ij}^{(t)} + \overline{(Q_{ij}^{(t)})}' Q_{ij}^{(s)} = 2\delta_{s,t} I^{(m_j)},$$

其中 $1 \le s, t \le n_{ij}$, $1 \le i < j \le N$;

(7)

$$Q_{ij}^{(s)} Q_{jk}^{(t)} = \sum_r (e_r A_{ij}^{sk} e_t') Q_{ik}^{(r)},$$

其中 $1 \le s \le n_{ij}$, $1 \le t \le n_{jk}$, $1 \le i < j < k \le N$;

(8)

$$\overline{(Q_{ij}^{(s)})}' Q_{ik}^{(t)} = \sum_r (e_t A_{ij}^{sk} e_r') Q_{jk}^{(r)},$$

其中 $1 \le s \le n_{ij}$, $1 \le t \le n_{ik}$, $1 \le i < j < k \le N$.

记 $x \in R^n$, 其中

$$n = \sum_{1 \le i \le j \le N} n_{ij},$$

又 $n_{11} = \cdots = n_{NN} = 1$. 任取 $x \in \mathbb{R}^n$, 则 x 可记作

$$x = (r_1, x_2, r_2, \cdots, x_N, r_N), \quad x_j = (x_{1j}, \cdots, x_{j-1,j}),$$

其中 $r_1, \cdots, r_N \in \mathbb{R}$, $x_{ij} \in \mathbb{R}^{n_{ij}}$. 记

$$C_j(x) = \begin{pmatrix} r_j & x_{j,j+1} & \cdots & x_{jN} \\ x_{j,j+1}' & r_{j+1} I^{(n_{j,j+1})} & \cdots & R_{j+1,N}^{(j)}(x) \\ \vdots & & & \vdots \\ x_{jN}' & R_{j+1,N}^{(j)}(x)' & \cdots & r_N I^{(n_{jN})} \end{pmatrix} \tag{2.1}$$

其中 $1 \le j \le N$, 于是 $C_j(x)$ 为

$$n_j' = n_{jj} + n_{j,j+1} + \cdots + n_{jN}$$

阶实对称方阵，其中 $n_{jk} \times n_{jl}$ 阶矩阵

$$R_{kl}^{(j)}(x) = \sum_r e_r' x_{kl} (A_{jk}^{rl})', \quad 1 \le j < k < l \le N \tag{2.2}$$

记 $u \in \mathbb{C}^m$, $m = m_1 + \cdots + m_N$, $u = (u_1, \cdots, u_N)$, $u_j \in \mathbb{C}^{m_j}$, 则

$$R_j(u) = \begin{pmatrix} u_j \\ R_{j+1}^{(j)}(u) \\ \vdots \\ R_N^{(j)}(u) \end{pmatrix} \tag{2.3}$$

为 $n_j' \times m_j$ 阶矩阵，其中

$$R_k^{(j)}(u) = \sum_r e_r' u_k \overline{(Q_{jk}^{(r)})}', \ 1 \le j < k \le N \tag{2.4}$$

定义 2.2 给定 N 矩阵组 $\mathfrak{B} = \{ A_{ij}^{tk}, Q_{ij}^{(t)} \}$, 由 N 矩阵组 $\mathfrak{B}$ 定义的秩为 N 的正规 Siegel 域定义为

$$\{ (z,u) \in \mathbb{C}^n \times \mathbb{C}^m \mid ImC_j(z) - ReR_j(u)\overline{R_j(u)}' > 0, \quad 1 \le j \le N \},$$

记作 $D(v_N, F)$.

定义 2.3 给定 $1 \le i < j \le N$, 并设 $n_{ij} > 0$, 称 $N \times N$ 对称方阵

$$S = \begin{pmatrix} n_{11} & \cdots & n_{1N} \\ \vdots & & \vdots \\ n_{N1} & \cdots & n_{NN} \end{pmatrix}$$

关于指标 (i,j) 适合 $Z(i,j)$ 条件，如果

(i) 任取 $l \in \{ i+1, \cdots, j-1 \}$, 只要 $(n_{il}, n_{lj}) \neq 0$, 就有

$$\begin{aligned} &n_{il} = n_{lj} = n_{ij}, \\ &n_{pi} = n_{pl} = n_{pj}, \quad p < i, \\ &n_{ip} = n_{lp} = n_{jp}, \quad j < p; \end{aligned}$$

(ii) 当 $p \in \{ i+1, \cdots, j-1 \}$ 时有

$$n_{ip} = n_{pl} = n_{pj}, \quad i < p < l; \quad n_{ip} = n_{lp} = n_{pj}, \quad l < p < j.$$

引理 2.4 [16] 设 $n_{ij} > 0$, 且矩阵 S 关于指标 (i,j) 适合定义 2.3 中的条件，则有

$$\begin{aligned} &\sum_t (A_{pi}^{tj})'(e_u'e_v + e_v'e_u)A_{pi}^{tj} = 2\delta_{uv} I^{(n_{ij})}, \\ &\sum_t A_{pi}^{tj}(e_u'e_v + e_v'e_u)(A_{pi}^{tj})' = 2\delta_{uv} I^{(n_{pj})}, \quad p < i, \\ &\sum_t (A_{ip}^{tj})'(e_u'e_v + e_v'e_u)A_{ip}^{tj} = 2\delta_{uv} I^{(n_{pj})}, \\ &\sum_t A_{ip}^{tj}(e_u'e_v + e_v'e_u)(A_{ip}^{tj})' = 2\delta_{uv} I^{(n_{ij})}, \quad i < p < j, \end{aligned}$$

以及

$$\begin{aligned} &A_{ij}^{tl}(A_{ij}^{sl})' + A_{ij}^{sl}(A_{ij}^{tl})' = 2\delta_{st} I^{(n_{il})}, \\ &A_{kl}^{sj}(A_{il}^{tj})' = \sum_r (e_s A_{ki}^{rl} e_t') A_{kl}^{rj}, \quad k < i < l < j, \\ &A_{kj}^{sp}(A_{ij}^{tp})' = \sum_r (e_s A_{ki}^{rj} e_t') A_{ki}^{rp}, \quad k < i < j < p, \end{aligned}$$

$$Q_{kl}^{(s)}\overline{(Q_{il}^{(t)})}' = \sum_r (e_s A_{ki}^{rl} e_t') Q_{ki}^{(r)}, \quad k < i < l < j,$$

$$Q_{kj}^{(s)}\overline{(Q_{ij}^{(t)})}' = \sum_r (e_s A_{ki}^{rj} e_t') Q_{ki}^{(r)}, \quad k < i < j < p.$$

引理 2.5 [17] $\mathbb{R}^n$ 或 $\mathbb{C}^n$ 中的域 D 上的解析 (或全纯) 向量场 $X = \sum_{i=1}^{n} \xi(x)\frac{\partial}{\partial x_i} \in \mathrm{aut}\,(D)$ 决定的单参数子群 $\exp(tX)$ 为域 D 上的解析 (或全纯) 自同构 $y = f(x,t), \forall t \in \mathbb{R}^n$, 它是常微分方程组

$$\frac{dy(t)}{dt} = \xi(y(t))$$

的适合初值

$$y(0) = x$$

的唯一解析解，其中 $\xi(x) = (\xi_1(x), \cdots, \xi_n(x))$.

定义 2.6 指标 $1 \le i_1 < \cdots < i_\rho \le N$ 是适合下面条件的最大集，这些条件是

(1) 实对称方阵 S 关于指标 (i_σ, i_τ) 适合定义 2.3 中的条件，并且有

$$m_{i_\sigma} = m_{i_\mu} = m_{i_\tau}, \quad i_\sigma < i_\mu < i_\tau;$$

(2) 当 $i \neq i_1 \cdots i_{\sigma-1}, i < i_\sigma$ 时，实对称方阵 S 关于指标 (i, i_σ) 不适合定义 2.3 中的条件；

(3)

$$n_{i_\sigma i} = 0, \quad i_\sigma < i, \quad i \neq i_{\sigma+1}, \cdots, i_\rho;$$

(4)

$$\sum_r (Q_{ii_\sigma}^{(r)})'(e_u' e_v + e_v' e_u) Q_{ii_\sigma}^{(r)} = 0, \quad 1 \le u, v \le m_i, \quad 1 \le i \le i_\sigma.$$

引理 2.7[16] $\mathrm{Iso}\,(D(v_N, F))$ 是一固定点 $p = (\sqrt{-1}v_0, 0) \in D(v_N, F)$ 的迷向子群， $\mathrm{iso}_p(D(v_N, F))$ 为 $\mathrm{Iso}_p(D(v_N, F))$ 的李代数，则 $\mathrm{iso}_p(D(v_N, F))$ 有空间直接和分解

$$\mathrm{iso}_p(D(v_N, F)) = o(D(v_N, F)) + \widetilde{y}(D(v_N, F)) + \widetilde{L_1} + \widetilde{L_2}$$

其中

(i)

$$o(D(v_N, F)) = \{ \sum_{i<j} z_{ij} L_{ij} \frac{\partial'}{\partial z_{ij}} + \sum_i u_i K_i \frac{\partial'}{\partial u_i} \}$$

是一个子代数， L_{ij} 是一个 $n_{ij} \times n_{ij}$ 阶实反对称矩阵满足

$$L_{ik}' A_{ij}^{tk} + A_{ij}^{tk} L_{jk} = \sum_r (e_r L_{ij} e_t') A_{ij}^{rk},$$

K_i 是一个 $m_i \times m_i$ 阶反 Hermitian 矩阵满足

$$\overline{K}_i' Q_{ij}^{(t)} + Q_{ij}^{(t)} K_j = \sum_r (e_r L_{ij} e_t') Q_{ij}^{(r)},$$

(ii) $\widetilde{y}(D(v_N,F))$ 是一个子空间，有基

$$X_{ij}^{(t)} - Z_{ij}^{(t)}, \quad 1 \le t \le n_{ij}, \quad X_{ij}^{(t)}, \quad Z_{ij}^{(t)} \in L_0,$$

(iii) $\widetilde{L_1}$ 是一个子空间，有基

$$Y_{i_\sigma}^{(t)} - \widetilde{P}_{i_\sigma}^{(t)}, \quad \widetilde{Y}_{i_\sigma}^{(t)} + P_{i_\sigma}^{(t)}, \quad 1 \le t \le m_{i_\sigma}, \quad 1 \le \sigma \le \rho,$$

$i_{\sigma_1}, \cdots, i_{\sigma_\rho}$ 满足定义 2.6.

(iv) $\widetilde{L_2}$ 是一个子空间，有基

$$B_{i_\sigma} + \frac{\partial}{\partial s_{i_\sigma}}, \quad T_{i_\sigma i_\tau}^{(t)} + 2\frac{\partial}{\partial z_{i_\sigma i_\tau}^{(t)}}, \quad 1 \le t \le n_{i_\sigma i_\tau}, \quad 1 \le \sigma < \tau \le \rho,$$

$i_{\sigma_1}, \cdots, i_{\sigma_\rho}$ 满足定义 2.6.
其中

$$\begin{aligned} X_{ij}^{(t)} - Z_{ij}^{(t)} = {} & 2z_{ij}^{(t)}\frac{\partial}{\partial s_i} - 2z_{ij}^{(t)}\frac{\partial}{\partial s_j} + (s_j - s_i)\frac{\partial}{\partial z_{ij}^{(t)}} + \sum_{p<i}\sum_{s} z_{pj}A_{pi}^{sj}e_t'\frac{\partial}{\partial z_{pi}^{(s)}} \\ & - \sum_{p<i}\sum_{s} z_{pi}^{(s)}e_t(A_{pi}^{sj})'\frac{\partial'}{\partial z_{pj}} + \sum_{i<p<j}\sum_{s} e_tA_{ip}^{sj}z_{pj}'\frac{\partial}{\partial z_{ip}^{(s)}} - \sum_{i<p<j}\sum_{s} z_{ip}^{(s)}e_tA_{ip}^{sj}\frac{\partial'}{\partial z_{pj}} \\ & + \sum_{p>j} z_{jp}(A_{ij}^{tp})'\frac{\partial'}{\partial z_{ip}} - \sum_{p>j} z_{ip}A_{ij}^{tp}\frac{\partial'}{\partial z_{jp}} + u_j\overline{(Q_{ij}^{(t)})}'\frac{\partial'}{\partial u_i} - u_iQ_{ij}^{(t)}\frac{\partial'}{\partial u_j}; \end{aligned}$$

$$\begin{aligned} Y_i^{(t)} - \widetilde{P}_i^{(t)} = {} & 2(\sqrt{-1}u_i^{(t)}s_i - u_i^{(t)})\frac{\partial}{\partial s_i} + 2\sqrt{-1}\sum_{p<i}\sum_{s} z_{pi}^{(s)}u_pQ_{pi}^{(s)}e_t'\frac{\partial}{\partial s_p} \\ & + 2\sqrt{-1}\sum_{i<p}\sum_{s} z_{ip}^{(s)}u_p\overline{(Q_{ip}^{(s)})}'e_t'\frac{\partial}{\partial s_p} + \sqrt{-1}\sum_{p<i}\sum_{s} u_pQ_{pi}^{(s)}e_t'[s_i\frac{\partial}{\partial z_{pi}^{(s)}} + \sqrt{-1}\frac{\partial}{\partial z_{pi}^{(s)}} \\ & + \sum_{l<p}\sum_{u} z_{li}A_{lp}^{ui}e_s'\frac{\partial}{\partial z_{lp}^{(u)}} + \sum_{p<l<i}\sum_{u} e_sA_{pl}^{ui}z_{li}'\frac{\partial}{\partial z_{pl}^{(u)}} + \sum_{i<l} z_{il}(A_{pi}^{sl})'\frac{\partial'}{\partial z_{pl}}] \\ & + \sqrt{-1}\sum_{p<i}\sum_{s,r} z_{pi}^{(s)}u_i\overline{(Q_{pi}^{(r)})}'Q_{pi}^{(s)}e_t'\frac{\partial}{\partial z_{pi}^{(r)}} + \sqrt{-1}\sum_{i<p}\sum_{s,r} z_{ip}^{(s)}u_iQ_{ip}^{(r)}\overline{(Q_{ip}^{(s)})}'e_t'\frac{\partial}{\partial z_{ip}^{(r)}} \\ & + \sqrt{-1}\sum_{i<p}\sum_{s} u_p\overline{(Q_{ip}^{(s)})}'e_t'[s_i\frac{\partial}{\partial z_{ip}^{(s)}} + \sqrt{-1}\frac{\partial}{\partial z_{ip}^{(s)}} + \sum_{l<i}\sum_{u} z_{li}^{(u)}e_s(A_{li}^{up})'\frac{\partial'}{\partial z_{lp}} \\ & + \sum_{i<l<p}\sum_{u} z_{il}^{(u)}e_sA_{il}^{up}\frac{\partial'}{\partial z_{lp}} + \sum_{p<l} z_{il}A_{ip}^{sl}\frac{\partial'}{\partial z_{pl}}] + \sum_{p<i}\sum_{s} z_{pi}^{(s)}e_t\overline{(Q_{pi}^{(s)})}'\frac{\partial'}{\partial u_p} \end{aligned}$$

$$+\sqrt{-1}\sum_{p<i}\sum_{s} u_p Q_{pi}^{(s)} e_t' u_i \overline{(Q_{pi}^{(s)})}' \frac{\partial'}{\partial u_p} + (2\sqrt{-1}u_i^{(t)} u_i + s_i e_t - \sqrt{-1}e_t)\frac{\partial'}{\partial u_i}$$

$$+\sum_{i<p}\sum_{s} z_{ip}^{(s)} e_t Q_{ip}^{(s)} \frac{\partial'}{\partial u_p} + \sqrt{-1}\sum_{i<p}\sum_{s} u_p \overline{(Q_{ip}^{(s)})}' e_t' u_i Q_{ip}^{(s)} \frac{\partial'}{\partial u_p}, \quad i \in \{ i_1, \cdots, i_\rho \},$$

$$\begin{aligned}
\tilde{Y}_i^{(t)} + P_i^{(t)} = {} & 2(u_i^{(t)} s_i + \sqrt{-1}u_i^{(t)})\frac{\partial}{\partial s_i} + 2\sum_{p<i}\sum_{s} z_{pi}^{(s)} u_p Q_{pi}^{(s)} e_t' \frac{\partial}{\partial s_p} \\
& + 2\sum_{i<p}\sum_{s} z_{ip}^{(s)} u_p \overline{(Q_{ip}^{(s)})}' e_t' \frac{\partial}{\partial s_p} + \sum_{p<i}\sum_{s} u_p Q_{pi}^{(s)} e_t' [s_i \frac{\partial}{\partial z_{pi}^{(s)}} + \sqrt{-1}\frac{\partial}{\partial z_{pi}^{(s)}} \\
& + \sum_{l<p}\sum_{u} z_{li} A_{lp}^{ui} e_s' \frac{\partial}{\partial z_{lp}^{(u)}} + \sum_{p<l<i}\sum_{u} e_s A_{pl}^{ui} z_{li}' \frac{\partial}{\partial z_{pl}^{(u)}} + \sum_{i<l} z_{il}(A_{pi}^{sl})' \frac{\partial'}{\partial z_{pl}}] \\
& + \sum_{p<i}\sum_{s,r} z_{pi}^{(s)} u_i \overline{(Q_{pi}^{(r)})}' Q_{pi}^{(s)} e_t' \frac{\partial}{\partial z_{pi}^{(r)}} + \sum_{i<p}\sum_{s,r} z_{ip}^{(s)} u_i Q_{ip}^{(r)} \overline{(Q_{ip}^{(s)})}' e_t' \frac{\partial}{\partial z_{ip}^{(r)}} \\
& + \sum_{i<p}\sum_{s} u_p \overline{(Q_{ip}^{(s)})}' e_t' [s_i \frac{\partial}{\partial z_{ip}^{(s)}} + \sqrt{-1}\frac{\partial}{\partial z_{ip}^{(s)}} + \sum_{l<i}\sum_{u} z_{li}^{(u)} e_s (A_{li}^{up})' \frac{\partial'}{\partial z_{lp}} \\
& + \sum_{i<l<p}\sum_{u} z_{il}^{(u)} e_s A_{il}^{up} \frac{\partial'}{\partial z_{lp}} + \sum_{p<l} z_{il} A_{ip}^{sl} \frac{\partial'}{\partial z_{pl}}] \\
& + \sqrt{-1}\sum_{p<i}\sum_{s} z_{pi}^{(s)} e_t \overline{(Q_{pi}^{(s)})}' \frac{\partial'}{\partial u_p} + \sum_{p<i}\sum_{s} u_p Q_{pi}^{(s)} e_t' u_i \overline{(Q_{pi}^{(s)})}' \frac{\partial'}{\partial u_p} \\
& + (2u_i^{(t)} u_i + \sqrt{-1}s_i e_t + e_t)\frac{\partial'}{\partial u_i} + \sqrt{-1}\sum_{i<p}\sum_{s} z_{ip}^{(s)} e_t Q_{ip}^{(s)} \frac{\partial'}{\partial u_p} \\
& + \sum_{i<p}\sum_{s} u_p \overline{(Q_{ip}^{(s)})}' e_t' u_i Q_{ip}^{(s)} \frac{\partial'}{\partial u_p}, \quad i \in \{ i_1, \cdots, i_\rho \};
\end{aligned}$$

$$\begin{aligned}
B_i + \frac{\partial}{\partial s_i} = {} & \sum_{p<i} z_{pi} z_{pi}' \frac{\partial}{\partial s_p} + \sum_{p>i} z_{ip} z_{ip}' \frac{\partial}{\partial s_p} + s_i^2 \frac{\partial}{\partial s_i} + s_i \sum_{p<i} z_{pi} \frac{\partial'}{\partial z_{pi}} + s_i \sum_{p>i} z_{ip} \frac{\partial'}{\partial z_{ip}} \\
& + \sum_{k<p<i}\sum_{s} z_{ki} A_{kp}^{si} z_{pi}' \frac{\partial}{\partial z_{kp}^{(s)}} + \sum_{k<i<p}\sum_{s} z_{ki}^{(s)} z_{ip} (A_{ki}^{sp})' \frac{\partial'}{\partial z_{kp}} \\
& + \sum_{i<k<p}\sum_{s} z_{ik}^{(s)} z_{ip} A_{ik}^{sp} \frac{\partial'}{\partial z_{kp}} + s_i u_i \frac{\partial'}{\partial u_i} + \sum_{p<i}\sum_{s} z_{pi}^{(s)} u_i \overline{(Q_{pi}^{(s)})}' \frac{\partial'}{\partial u_p} \\
& + \sum_{p>i}\sum_{s} z_{ip}^{(s)} u_i Q_{ip}^{(s)} \frac{\partial'}{\partial u_p} + \frac{\partial}{\partial s_i}, \quad i \in \{ i_1, \cdots, i_\rho \},
\end{aligned}$$

$$
\begin{aligned}
T_{ij}^{(t)} + 2\frac{\partial}{\partial z_{ij}^{(t)}} = {} & 2s_i z_{ij}^{(t)} \frac{\partial}{\partial s_i} + 2s_j z_{ij}^{(t)} \frac{\partial}{\partial s_j} + 2\sum_{p<i}\sum_{s} z_{pi}^{(s)} z_{pj} A_{pi}^{sj} e_t' \frac{\partial}{\partial s_p} \\
& + 2\sum_{i<p<j}\sum_{s} z_{ip}^{(s)} e_t A_{ip}^{sj} z_{pj}' \frac{\partial}{\partial s_p} + 2\sum_{p>j} z_{ip} A_{ij}^{tp} z_{jp}' \frac{\partial}{\partial s_p} \\
& + \sum_{p<i}[2z_{ij}^{(t)} z_{pi} + s_i \sum_{s} z_{pj} A_{pi}^{sj} e_t' e_s - \sum_{s,r} z_{pi}^{(s)} z_{ij} (A_{pi}^{sj})' A_{pi}^{rj} e_t' e_r] \frac{\partial'}{\partial z_{pi}} \\
& + \sum_{i<p}[2z_{ij}^{(t)} z_{ip} + s_i \sum_{p<j}\sum_{s} e_t A_{ip}^{sj} z_{pj}' e_s - \sum_{p<j}\sum_{s,r} z_{ip}^{(s)} z_{ij} A_{ip}^{sj} (A_{ip}^{rj})' e_t' e_r \\
& + s_i \sum_{p>j} z_{jp} (A_{ij}^{tp})' - \sum_{p>j}\sum_{s} z_{ij}^{(s)} z_{ip} A_{ij}^{sp} (A_{ij}^{tp})'] \frac{\partial'}{\partial z_{ip}} + \sum_{p>j}[s_j z_{ip} A_{ij}^{tp} \frac{\partial'}{\partial z_{jp}} \\
& + \sum_{i<k<p}\sum_{s} z_{ik}^{(s)} z_{jp} (A_{ij}^{tp})' A_{ik}^{sp} \frac{\partial'}{\partial z_{kp}}] + (s_i s_j - z_{ij} z_{ij}') \frac{\partial}{\partial z_{ij}^{(t)}} \\
& + \sum_{p<k<i}\sum_{s,r} z_{pj} A_{pi}^{sj} e_t' e_s A_{pk}^{ri} z_{ki}' \frac{\partial}{\partial z_{pk}^{(r)}} + \sum_{k<p<i}\sum_{s,r} z_{pj} A_{pi}^{sj} e_t' e_s (A_{kp}^{ri})' z_{ki}' \frac{\partial}{\partial z_{kp}^{(r)}} \\
& + \sum_{p<i<k}\sum_{s} z_{pj} A_{pi}^{sj} e_t' z_{ik} (A_{pi}^{sk})' \frac{\partial'}{\partial z_{pk}} + \sum_{k<i<p<j}\sum_{s,r} z_{ki}^{(r)} e_t A_{ip}^{sj} z_{pj}' e_s (A_{ki}^{rp})' \frac{\partial'}{\partial z_{kp}} \\
& + s_j \sum_{k<i}\sum_{s} z_{ki}^{(s)} e_t (A_{ki}^{sj})' \frac{\partial'}{\partial z_{kj}} + \sum_{k<i<j<p}\sum_{s} z_{ki}^{(s)} z_{jp} (A_{ij}^{tp})' (A_{ki}^{sp})' \frac{\partial'}{\partial z_{kp}} \\
& + \sum_{i<p<k,p<j}\sum_{s} e_t A_{ip}^{sj} z_{pj}' z_{ik} A_{ip}^{sk} \frac{\partial'}{\partial z_{pk}} + s_j \sum_{i<k<j}\sum_{s} z_{ik}^{(s)} e_t A_{ik}^{sj} \frac{\partial'}{\partial z_{kj}} \\
& + \sum_{i<k<p<j}\sum_{s,r} e_t A_{ip}^{sj} z_{pj}' z_{ik}^{(r)} e_s A_{ik}^{rp} \frac{\partial'}{\partial z_{kp}} + \sum_{j<p<k}\sum_{s} z_{jp} (A_{ij}^{tp})' e_s' z_{ik} A_{ip}^{sk} \frac{\partial'}{\partial z_{pk}} \\
& + s_j u_i Q_{ij}^{(t)} \frac{\partial'}{\partial u_j} + \sum_{p<i}\sum_{s} z_{pj} A_{pi}^{sj} e_t' u_i \overline{(Q_{pi}^{(s)})}' \frac{\partial'}{\partial u_p} + \sum_{i<p<j}\sum_{s} e_t A_{ip}^{sj} z_{pj}' u_i Q_{ip}^{(s)} \frac{\partial'}{\partial u_p} \\
& + \sum_{p>j}\sum_{s} z_{jp} (A_{ij}^{tp})' e_s' u_i Q_{ip}^{(s)} \frac{\partial'}{\partial u_p} + \sum_{i<p}\sum_{s} z_{ip}^{(s)} u_j \overline{(Q_{ij}^{(t)})}' Q_{ip}^{(s)} \frac{\partial'}{\partial u_p} \\
& + \sum_{p<i}\sum_{s} z_{pi}^{(s)} u_j \overline{(Q_{ij}^{(t)})}' \overline{(Q_{pi}^{(s)})}' \frac{\partial'}{\partial u_p} + (2z_{ij}^{(t)} u_i - \sum_{s} z_{ij}^{(s)} u_i Q_{ij}^{(s)} \overline{(Q_{ij}^{(t)})}' \\
& + u_j \overline{(Q_{ij}^{(t)})}' s_i) \frac{\partial'}{\partial u_i} + 2\frac{\partial}{\partial z_{ij}^{(t)}}, \quad 1 \le t \le n_{ij},\ i<j,\ i,j \in \{\, i_1, \cdots, i_\rho \,\}.
\end{aligned}
$$

第三章　正规 Siegel 域的迷向子群

本章第一节为许以超教授已做工作：给出了紧李子群 $\exp(o(D(v_N,F)))$, 后三节我们通过求解一些常微分方程组，求出 $\widetilde{y}(D(v_N,F))$, $\widetilde{L_1}$, $\widetilde{L_2}$ 中基元所决定的单参数子群，进而给出正规 Siegel 域一固定点 $(\sqrt{-1}v_0,0)$ 的迷向子群的生成元集，即证明了第一章中的定理.

不妨设向量 $z=(s_1,z_2,s_2,\cdots,z_N,s_N)$ 经全纯自同构 σ 作用后变为向量 $w=(r_1,w_2,r_2,\cdots,w_N,r_N)$, 向量 $u=(u_1,\cdots,u_N)$ 变为向量 $v=(v_1,\cdots,v_N)$.

§3.1　$\exp(o(D(v_N,F)))$

$\exp(o(D(v_N,F))$ 可以表示为：

$$
\begin{aligned}
&r_j=s_j,\quad 1\le j\le N,\\
&w_{ij}=z_{ij}O_{ij},\quad 1\le i<j\le N,\\
&v_i=u_iU_i,\quad 1\le i\le N,
\end{aligned}
$$

其中 $O_{ij}\in O(n_{ij})$, $U_i\in U(m_i)$, 且满足

$$O'_{ik}A^{tk}_{ij}O_{jk}=\sum_{r=1}^{n_{ij}}(e_rO_{ij}e'_t)A^{rk}_{ij},\quad \overline{U_i}'Q^{(t)}_{ij}U_j=\sum_{r=1}^{n_{ij}}(e_rO_{ij}e'_t)Q^{(r)}_{ij}.$$

§3.2　$\exp(\theta(X^{(t)}_{ij}-Z^{(t)}_{ij}))$

在这一节，我们计算 $X^{(t)}_{ij}-Z^{(t)}_{ij}$ 所决定的单参数子群.

由引理 2.5 列出满足零点初值为 (z,u) 的微分方程组

$$
\begin{aligned}
&\frac{dr_i}{d\theta}=2w^{(t)}_{ij},\\
&\frac{dr_j}{d\theta}=-2w^{(t)}_{ij},\\
&\frac{dw^{(t)}_{ij}}{d\theta}=r_j-r_i,\\
&\frac{dw^{(s)}_{pi}}{d\theta}=w_{pj}A^{sj}_{pi}e'_t,\quad p<i,\\
&\frac{dw_{pj}}{d\theta}=-\sum_s w^{(s)}_{pi}e_t(A^{sj}_{pi})',\quad p<i,\\
&\frac{dw^{(s)}_{ip}}{d\theta}=e_tA^{sj}_{ip}w'_{pj},\quad i<p<j\\
&\frac{dw_{pj}}{d\theta}=-\sum_s w^{(s)}_{ip}e_tA^{sj}_{ip},\quad i<p<j,\\
&\frac{dw_{ip}}{d\theta}=w_{jp}(A^{tp}_{ij})',\quad p>j,\\
&\frac{dw_{jp}}{d\theta}=-w_{ip}A^{tp}_{ij},\quad p>j,
\end{aligned}
$$

$$\frac{dv_i}{d\theta} = v_j\overline{(Q_{ij}^{(t)})}',$$
$$\frac{dv_j}{d\theta} = -v_i Q_{ij}^{(t)},$$
$$\frac{dv_p}{d\theta} = 0, \quad p \neq i, j.$$

下面求解这个常微分方程组.

(1) 由 $\left\{\begin{array}{l}\frac{dv_p}{d\theta} = 0,\ p \neq i,j \\ v_p(0) = u_p\end{array}\right.$，可得 $v_p = u_p$.

(2)

由 $\left\{\begin{array}{l}\frac{dv_i}{d\theta} = v_j\overline{(Q_{ij}^{(t)})}' \\ \frac{dv_j}{d\theta} = -v_i Q_{ij}^{(t)}\end{array}\right.$ 可得 $\frac{d}{d\theta}\begin{pmatrix} v_i' \\ v_j' \end{pmatrix} = \begin{pmatrix} 0 & \overline{Q_{ij}^{(t)}} \\ -Q_{ij}^{(t)\prime} & 0 \end{pmatrix}\begin{pmatrix} v_i' \\ v_j' \end{pmatrix}$,

该常系数齐线性方程组满足初值条件 $v_i(0) = u_i$, $v_j(0) = u_j$ 的解为

$$(v_i, v_j) = (u_i, u_j)\exp\theta\begin{pmatrix} 0 & -Q_{ij}^{(t)} \\ \overline{Q_{ij}^{(t)}}' & 0 \end{pmatrix}.$$

令

$$A = \begin{pmatrix} 0 & -Q_{ij}^{(t)} \\ \overline{(Q_{ij}^{(t)})}' & 0 \end{pmatrix},$$

则

$$A^2 = \begin{pmatrix} -Q_{ij}^{(t)}\overline{(Q_{ij}^{(t)})}' & 0 \\ 0 & -\overline{(Q_{ij}^{(t)})}' Q_{ij}^{(t)} \end{pmatrix} = \begin{pmatrix} -E & 0 \\ 0 & -E \end{pmatrix},$$

$$A^{2n} = (-1)^n\begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix}, \quad A^{2n+1} = (-1)^n\begin{pmatrix} 0 & -Q_{ij}^{(t)} \\ \overline{(Q_{ij}^{(t)})}' & 0 \end{pmatrix},$$

记

$$\exp\theta A = \begin{pmatrix} A_{11}(\theta) & A_{12}(\theta) \\ A_{21}(\theta) & A_{22}(\theta) \end{pmatrix},$$

则

$$\begin{pmatrix} A_{11}(\theta) & 0 \\ 0 & A_{22}(\theta) \end{pmatrix} = E + \sum_{n=1}^{\infty}\frac{(-1)^n\theta^{2n}}{(2n)!}\begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} = \sum_{n=0}^{\infty}\frac{(-1)^n\theta^{2n}}{(2n)!}E = \cos\theta E,$$

$$\begin{pmatrix} 0 & A_{12}(\theta) \\ A_{21}(\theta) & 0 \end{pmatrix} = \sum_{n=0}^{\infty}\frac{(-1)^n\theta^{2n+1}}{(2n+1)!}\begin{pmatrix} 0 & -Q_{ij}^{(t)} \\ \overline{(Q_{ij}^{(t)})}' & 0 \end{pmatrix} = \sin\theta\begin{pmatrix} 0 & -Q_{ij}^{(t)} \\ \overline{(Q_{ij}^{(t)})}' & 0 \end{pmatrix},$$

所以

$$(v_i, v_j) = (u_i, u_j)\exp\theta A = (u_i\cos\theta + u_j\overline{(Q_{ij}^{(t)})}'\sin\theta, -u_i Q_{ij}^{(t)}\sin\theta + u_j\cos\theta).$$

即

$$v_i = u_i\cos\theta + u_j\overline{(Q_{ij}^{(t)})'}\sin\theta,$$
$$v_j = -u_iQ_{ij}^{(t)}\sin\theta + u_j\cos\theta.$$

(3)

由 $\begin{cases}\frac{dr_i}{d\theta} = 2w_{ij}^{(t)}\\ \frac{dr_j}{d\theta} = -2w_{ij}^{(t)}\\ \frac{dw_{ij}^{(t)}}{d\theta} = r_j - r_i\end{cases}$ 可得 $\frac{d}{d\theta}\begin{pmatrix} r_i\\ r_j\\ w_{ij}^{(t)}\end{pmatrix} = \begin{pmatrix} 0 & 0 & 2\\ 0 & 0 & -2\\ -1 & 1 & 0\end{pmatrix}\begin{pmatrix} r_i\\ r_j\\ w_{ij}^{(t)}\end{pmatrix}$,

该常系数齐线性方程组满足初值条件 $r_i(0) = s_i$, $r_j(0) = s_j$, $w_{ij}^{(t)}(0) = z_{ij}^{(t)}$ 的解为

$$(r_i, r_j, w_{ij}^{(t)}) = (s_i, s_j, z_{ij}^{(t)})\exp\theta\begin{pmatrix} 0 & 0 & -1\\ 0 & 0 & 1\\ 2 & -2 & 0\end{pmatrix}.$$

令

$$A = \begin{pmatrix} 0 & 0 & -1\\ 0 & 0 & 1\\ 2 & -2 & 0\end{pmatrix},$$

则

$$A^2 = \begin{pmatrix} -2 & 2 & 0\\ 2 & -2 & 0\\ 0 & 0 & -4\end{pmatrix} = 2\begin{pmatrix} -1 & 1 & 0\\ 1 & -1 & 0\\ 0 & 0 & -2\end{pmatrix},$$

因为:

$$\begin{pmatrix} -1 & 1\\ 1 & -1\end{pmatrix}^n = (-2)^{n-1}\begin{pmatrix} -1 & 1\\ 1 & -1\end{pmatrix},$$

所以:

$$A^{2n} = (-1)^n 2^{2n-1}\begin{pmatrix} 1 & -1 & 0\\ -1 & 1 & 0\\ 0 & 0 & 2\end{pmatrix}, \quad A^{2n+1} = (-1)^n 2^{2n}\begin{pmatrix} 0 & 0 & -1\\ 0 & 0 & 1\\ 2 & -2 & 0\end{pmatrix},$$

记

$$\exp\theta A = \begin{pmatrix} A_{11}(\theta) & A_{12}(\theta) & A_{13}(\theta)\\ A_{21}(\theta) & A_{22}(\theta) & A_{23}(\theta)\\ A_{31}(\theta) & A_{32}(\theta) & A_{33}(\theta)\end{pmatrix}.$$

则

$$\begin{pmatrix} 0 & 0 & A_{13}(\theta)\\ 0 & 0 & A_{23}(\theta)\\ A_{31}(\theta) & A_{32}(\theta) & 0\end{pmatrix} = \sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n}\theta^{2n+1}}{(2n+1)!}\begin{pmatrix} 0 & 0 & -1\\ 0 & 0 & 1\\ 2 & -2 & 0\end{pmatrix} = \frac{\sin 2\theta}{2}\begin{pmatrix} 0 & 0 & -1\\ 0 & 0 & 1\\ 2 & -2 & 0\end{pmatrix},$$

$$
\begin{pmatrix} A_{11}(\theta) & A_{12}(\theta) & 0 \\ A_{21}(\theta) & A_{22}(\theta) & 0 \\ 0 & 0 & A_{33}(\theta) \end{pmatrix} = E + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n-1}\theta^{2n}}{(2n)!} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}
$$

$$
= \frac{1}{2}\cos 2\theta \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}
$$

$$
= \begin{pmatrix} \frac{1}{2}\cos 2\theta + \frac{1}{2} & -\frac{1}{2}\cos 2\theta + \frac{1}{2} & 0 \\ -\frac{1}{2}\cos 2\theta + \frac{1}{2} & \frac{1}{2}\cos 2\theta + \frac{1}{2} & 0 \\ 0 & 0 & \cos 2\theta \end{pmatrix},
$$

所以：

$$
(r_i, r_j, w_{ij}^{(t)}) = (s_i, s_j, z_{ij}^{(t)}) \begin{pmatrix} \frac{1}{2}\cos 2\theta + \frac{1}{2} & -\frac{1}{2}\cos 2\theta + \frac{1}{2} & -\frac{1}{2}\sin 2\theta \\ -\frac{1}{2}\cos 2\theta + \frac{1}{2} & \frac{1}{2}\cos 2\theta + \frac{1}{2} & \frac{1}{2}\sin 2\theta \\ \sin 2\theta & -\sin 2\theta & \cos 2\theta \end{pmatrix},
$$

即：

$$
r_i = (\frac{1}{2}\cos 2\theta + \frac{1}{2})s_i + (-\frac{1}{2}\cos 2\theta + \frac{1}{2})s_j + \sin 2\theta z_{ij}^{(t)},
$$

$$
r_j = (-\frac{1}{2}\cos 2\theta + \frac{1}{2})s_i + (\frac{1}{2}\cos 2\theta + \frac{1}{2})s_j - \sin 2\theta z_{ij}^{(t)},
$$

$$
w_{ij}^{(t)} = -\frac{1}{2}\sin 2\theta s_i + \frac{1}{2}\sin 2\theta s_j + \cos 2\theta z_{ij}^{(t)}.
$$

(4) $j < p$ 时，同 (2) 的讨论.

$\begin{cases} \frac{dw_{ip}}{d\theta} = w_{jp}(A_{ij}^{tp})' \\ \frac{dw_{jp}}{d\theta} = -w_{ip}A_{ij}^{tp} \end{cases}$ 满足初值条件 $w_{ip}(0) = z_{ip}, w_{jp}(0) = z_{jp}$ 的解为

$$
w_{ip} = z_{ip}\cos\theta + z_{jp}(A_{ij}^{tp})'\sin\theta,
$$

$$
w_{jp} = -z_{ip}A_{ij}^{tp}\sin\theta + z_{jp}\cos\theta.
$$

(5) $p < i$ 时

由 $\begin{cases} \frac{dw_{pi}^{(s)}}{d\theta} = w_{pj}A_{pi}^{sj}e_t', \quad s = 1, \cdots, n_{pi}, \\ \frac{dw_{pj}}{d\theta} = -\sum\limits_s w_{pi}^{(s)} e_t (A_{pi}^{sj})' \end{cases}$ 可得

$$
\frac{d}{d\theta} \begin{pmatrix} w_{pi}^{(1)} \\ \vdots \\ w_{pi}^{(n_{pi})} \\ w_{pj}' \end{pmatrix} = \begin{pmatrix} 0 & \cdots & 0 & e_t(A_{pi}^{1j})' \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & e_t(A_{pi}^{n_{pi}j})' \\ -A_{pi}^{1j}e_t' & \cdots & -A_{pi}^{n_{pi}j}e_t' & 0 \end{pmatrix} \begin{pmatrix} w_{pi}^{(1)} \\ \vdots \\ w_{pi}^{(n_{pi})} \\ w_{pj}' \end{pmatrix},
$$

该常系数齐线性方程组满足初值条件 $(w_{pi}, w_{pj})(0) = (z_{pi}, z_{pj})$ 的解为

$$(w_{pi}^{(1)}, \cdots, w_{pi}^{(n_{pi})}, w_{pj}) = (z_{pi}^{(1)}, \cdots, z_{pi}^{(n_{pi})}, z_{pj}) \exp\theta \begin{pmatrix} 0 & \cdots & 0 & -e_t(A_{pi}^{1j})' \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & -e_t(A_{pi}^{n_{pi}j})' \\ A_{pi}^{1j}e_t' & \cdots & A_{pi}^{n_{pi}j}e_t' & 0 \end{pmatrix}.$$

令

$$A = \begin{pmatrix} 0 & \cdots & 0 & -e_t(A_{pi}^{1j})' \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & -e_t(A_{pi}^{n_{pi}j})' \\ A_{pi}^{1j}e_t' & \cdots & A_{pi}^{n_{pi}j}e_t' & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} e_t(A_{pi}^{1j})' \\ \vdots \\ e_t(A_{pi}^{n_{pi}j})' \end{pmatrix} = \sum_{s=1}^{n_{pi}} e_s' e_t (A_{pi}^{sj})'.$$

则

$$A = \begin{pmatrix} 0 & -\beta \\ \beta' & 0 \end{pmatrix},$$

$$\beta\beta' = \sum_{s=1}^{n_{pi}}\sum_{r=1}^{n_{pi}} e_s' e_t (A_{pi}^{sj})' A_{pi}^{rj} e_t' e_r = E, \quad \beta'\beta = \sum_{s=1}^{n_{pi}} A_{pi}^{sj} E_{tt} (A_{pi}^{sj})' = E.$$

$$A^2 = \begin{pmatrix} -\beta\beta' & 0 \\ 0 & -\beta'\beta \end{pmatrix} = -E,$$

$$A^{2n} = (-1)^n E, \quad A^{2n+1} = (-1)^n \begin{pmatrix} 0 & -\beta \\ \beta' & 0 \end{pmatrix},$$

记

$$\exp\theta A = \begin{pmatrix} A_{11}(\theta) & A_{12}(\theta) \\ A_{21}(\theta) & A_{22}(\theta) \end{pmatrix},$$

则

$$\begin{pmatrix} A_{11}(\theta) & 0 \\ 0 & A_{22}(\theta) \end{pmatrix} = E + \sum_{n=1}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n)!} E = \cos\theta E,$$

$$\begin{pmatrix} 0 & A_{12}(\theta) \\ A_{21}(\theta) & 0 \end{pmatrix} = \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n+1}}{(2n+1)!} \begin{pmatrix} 0 & -\beta \\ \beta' & 0 \end{pmatrix} = \sin\theta \begin{pmatrix} 0 & -\beta \\ \beta' & 0 \end{pmatrix},$$

所以

$$(w_{pi}^{(1)}, \cdots, w_{pi}^{(n_{pi})}, w_{pj}) = (z_{pi}^{(1)}, \cdots, z_{pi}^{(n_{pi})}, z_{pj}) \begin{pmatrix} \cos\theta E & \sin\theta(-\beta) \\ \sin\theta\beta' & \cos\theta E \end{pmatrix}.$$

即

$$w_{pi} = z_{pi}\cos\theta + z_{pj}\sum_{s=1}^{n_{pi}} A_{pi}^{sj} e_t' e_s \sin\theta,$$

$$w_{pj} = -z_{pi}\sum_{s=1}^{n_{pi}} e_s' e_t (A_{pi}^{sj})' \sin\theta + z_{pj}\cos\theta.$$

(6) $i<p<j$ 时，同 (5) 的讨论．

$\begin{cases} \frac{dw_{ip}^{(s)}}{d\theta} = e_t A_{ip}^{sj} w_{pj}', \quad s=1,\cdots,n_{ip} \\ \frac{dw_{pj}}{d\theta} = -\sum_s w_{ip}^{(s)} e_t A_{ip}^{sj} \end{cases}$ 满足初值条件 $(w_{ip}, w_{pj})(0) = (z_{ip}, z_{pj})$ 的解为

$$w_{ip} = z_{ip}\cos\theta + +z_{pj}\sum_s (A_{ip}^{sj})' e_t' e_s \sin\theta,$$

$$w_{pj} = -z_{ip}\sum_s e_s' e_t A_{ip}^{sj} \sin\theta + z_{pj}\cos\theta.$$

所以， $\exp(\theta(X_{ij}^{(t)} - Z_{ij}^{(t)}))$ 可以表示为：

$$\begin{aligned}
& r_p = s_p, \quad p \neq i, j, \\
& r_i = (\frac{1}{2}\cos 2\theta + \frac{1}{2})s_i + (-\frac{1}{2}\cos 2\theta + \frac{1}{2})s_j + \sin 2\theta z_{ij}^{(t)}, \\
& r_j = (-\frac{1}{2}\cos 2\theta + \frac{1}{2})s_i + (\frac{1}{2}\cos 2\theta + \frac{1}{2})s_j - \sin 2\theta z_{ij}^{(t)}, \\
& w_{ij}^{(s)} = z_{ij}^{(s)}, \quad s \neq t, \\
& w_{ij}^{(t)} = -\frac{1}{2}\sin 2\theta s_i + \frac{1}{2}\sin 2\theta s_j + \cos 2\theta z_{ij}^{(t)}, \\
& w_{pi} = z_{pi}\cos\theta + z_{pj}\sum_s A_{pi}^{sj} e_t' e_s \sin\theta, \quad p < i, \\
& w_{pj} = -z_{pi}\sum_s e_s' e_t (A_{pi}^{sj})' \sin\theta + z_{pj}\cos\theta, \quad p < i, \\
& w_{ip} = z_{ip}\cos\theta + z_{pj}\sum_s (A_{ip}^{sj})' e_t' e_s \sin\theta, \quad i < p < j, \\
& w_{pj} = -z_{ip}\sum_s e_s' e_t A_{ip}^{sj} \sin\theta + z_{pj}\cos\theta, \quad i < p < j, \\
& w_{ip} = z_{ip}\cos\theta + z_{jp}(A_{ij}^{tp})' \sin\theta, \quad p > j, \\
& w_{jp} = -z_{ip} A_{ij}^{tp} \sin\theta + z_{jp}\cos\theta, \quad p > j, \\
& v_p = u_p, \quad p \neq i, j, \\
& v_i = u_i\cos\theta + u_j \overline{(Q_{ij}^{(t)})}' \sin\theta, \\
& v_j = -u_i Q_{ij}^{(t)} \sin\theta + u_j\cos\theta.
\end{aligned}$$

§3.3 $\exp(\theta(Y_i^{(t)} - \widetilde{P}_i^{(t)}))$, $\exp(\theta(\widetilde{Y}_i^{(t)} + P_i^{(t)}))$

在这一节，我们计算 $Y_i^{(t)}-\widetilde{P}_i^{(t)}, \widetilde{Y}_i^{(t)}+P_i^{(t)}$ 所决定的单参数子群.

一 计算 $Y_i^{(t)}-\widetilde{P}_i^{(t)},\quad i\in\{i_1,\cdots,i_\rho\}$ 所决定的单参数子群 $\exp(\theta(Y_i^{(t)}-\widetilde{P}_i^{(t)}))$.

由引理 2.5 列出满足零点初值为 (z,u) 的常微分方程组

$$
\begin{aligned}
&\frac{dr_i}{d\theta}=2\sqrt{-1}r_iv_i^{(t)}-2v_i^{(t)},\\
&\frac{dr_p}{d\theta}=2\sqrt{-1}\sum_s w_{pi}^{(s)}v_pQ_{pi}^{(s)}e_t',\quad p<i,\\
&\frac{dr_p}{d\theta}=2\sqrt{-1}\sum_s w_{ip}^{(s)}v_p\overline{(Q_{ip}^{(s)})}'e_t',\quad p>i,\\
&\frac{dv_i}{d\theta}=2\sqrt{-1}v_i^{(t)}v_i+r_ie_t-\sqrt{-1}e_t,\\
&\frac{dv_p}{d\theta}=\sum_s[w_{pi}^{(s)}e_t\overline{(Q_{pi}^{(s)})}'+\sqrt{-1}v_pQ_{pi}^{(s)}e_t'v_i\overline{(Q_{pi}^{(s)})}'],\quad p<i,\\
&\frac{dv_p}{d\theta}=\sum_s[w_{ip}^{(s)}e_tQ_{ip}^{(s)}+\sqrt{-1}v_p\overline{(Q_{ip}^{(s)})}'e_t'v_iQ_{ip}^{(s)}],\quad p>i,\\
&\frac{dw_{pi}}{d\theta}=\sqrt{-1}\sum_s(r_i+\sqrt{-1})v_pQ_{pi}^{(s)}e_t'e_s+\sqrt{-1}\sum_{s,r}w_{pi}^{(s)}v_i\overline{(Q_{pi}^{(r)})}'Q_{pi}^{(s)}e_t'e_r,\quad p<i,\\
&\frac{dw_{ip}}{d\theta}=\sqrt{-1}\sum_s(r_i+\sqrt{-1})v_p\overline{(Q_{ip}^{(s)})}'e_t'e_s+\sqrt{-1}\sum_{s,r}w_{ip}^{(s)}v_iQ_{ip}^{(r)}\overline{(Q_{ip}^{(s)})}'e_t'e_r,\quad p>i,\\
&\frac{dw_{lp}}{d\theta}=\sqrt{-1}\sum_{s,u}[v_pQ_{pi}^{(s)}e_t'w_{li}A_{lp}^{ui}e_s'e_u+v_lQ_{li}^{(s)}e_t'e_sA_{lp}^{ui}w_{pi}'e_u],\quad l<p<i,\\
&\frac{dw_{lp}}{d\theta}=\sqrt{-1}\sum_s v_lQ_{li}^{(s)}e_t'w_{ip}(A_{li}^{sp})'+\sqrt{-1}\sum_{s,u}w_{li}^{(u)}v_p\overline{(Q_{ip}^{(s)})}'e_t'e_s(A_{li}^{up})',\quad l<i<p,\\
&\frac{dw_{lp}}{d\theta}=\sqrt{-1}\sum_s v_l\overline{(Q_{il}^{(s)})}'e_t'w_{ip}A_{il}^{sp}+\sqrt{-1}\sum_{s,u}w_{il}^{(u)}v_p\overline{(Q_{ip}^{(s)})}'e_t'e_sA_{il}^{up},\quad i<l<p.
\end{aligned}
$$

下面求解这个常微分方程组.

(1) 先求 r_i.

由

$$\frac{dr_i}{d\theta}=2\sqrt{-1}r_iv_i^{(t)}-2v_i^{(t)},$$

得

$$\frac{d^2r_i}{d\theta^2}=2\sqrt{-1}\frac{dr_i}{d\theta}v_i^{(t)}+2\sqrt{-1}r_i\frac{dv_i^{(t)}}{d\theta}-2\frac{dv_i^{(t)}}{d\theta}.$$

由 $\frac{dr_i}{d\theta}=2\sqrt{-1}r_iv_i^{(t)}-2v_i^{(t)}$, 可得

$$v_i^{(t)}=\frac{\frac{dr_i}{d\theta}}{2\sqrt{-1}(r_i+\sqrt{-1})},$$

由 $\frac{dv_i}{d\theta}=2\sqrt{-1}v_i^{(t)}v_i+r_ie_t-\sqrt{-1}e_t$, 可得

$$\frac{dv_i^{(t)}}{d\theta}=2\sqrt{-1}v_i^{(t)2}+r_i-\sqrt{-1}.$$

所以可列出满足初值条件的微分方程：

$$\begin{cases}\frac{d^2r_i}{d\theta^2}=\frac{2}{r_i+\sqrt{-1}}(\frac{dr_i}{d\theta})^2+2\sqrt{-1}(r_i^2+1)\\ r_i(0)=s_i\\ \frac{dr_i}{d\theta}(0)=2\sqrt{-1}s_iu_i^{(t)}-2u_i^{(t)}\end{cases}$$

令 $p=\frac{dr_i}{d\theta}$, 并以 r_i 为自变量， p 为新的未知函数，可得：

$$p\frac{dp}{dr_i}=\frac{2}{r_i+\sqrt{-1}}p^2+2\sqrt{-1}(r_i+\sqrt{-1})(r_i-\sqrt{-1}),$$

令 $u=r_i+\sqrt{-1}$, 可得

$$p\frac{dp}{du}=\frac{2}{u}p^2+2\sqrt{-1}u(u-2\sqrt{-1}),$$

令 $q=\frac{p}{u^2}$, 可得：

$$qdq=\frac{2\sqrt{-1}}{u^2}du+\frac{4}{u^3}du,$$

两边积分得

$$q^2=-\frac{4\sqrt{-1}}{u}-\frac{4}{u^2}+c, \tag{3.3.1}$$

$$p=\frac{dr_i}{d\theta}=\frac{du}{d\theta},$$

所以，

$$\frac{du}{d\theta}=u^2\sqrt{c-\frac{4\sqrt{-1}}{u}-\frac{4}{u^2}},$$

$$\frac{-d\frac{1}{u}}{\sqrt{c-\frac{4\sqrt{-1}}{u}-\frac{4}{u^2}}}=d\theta.$$

令

$$x=\frac{1}{u}=\frac{1}{r_i+\sqrt{-1}},$$

则

$$\frac{dx}{\sqrt{c-4\sqrt{-1}x-4x^2}}=-d\theta,$$

$$\frac{\frac{1}{2}d\frac{2}{\sqrt{c-1}}(x+\frac{\sqrt{-1}}{2})}{\sqrt{1-[\frac{2}{\sqrt{c-1}}(x+\frac{\sqrt{-1}}{2})]^2}}=-d\theta,$$

$$arc\sin\frac{2}{\sqrt{c-1}}(x+\frac{\sqrt{-1}}{2})=-2\theta-c', \tag{3.3.2}$$

$$x=-\frac{\sqrt{c-1}}{2}\sin(2\theta+c')-\frac{\sqrt{-1}}{2},$$

所以

$$r_i=-\frac{2}{\sqrt{c-1}\sin(2\theta+c')+\sqrt{-1}}-\sqrt{-1}.$$

在 (3.3.1) 式和 (3.3.2) 式中，令 $\theta=0$, 得

$$c=\frac{4\sqrt{-1}s_i-4{u_i^{(t)}}^2}{(s_i+\sqrt{-1})^2},\qquad c'=\mathrm{arctg}\,\frac{\sqrt{-1}-s_i}{2u_i^{(t)}}.$$

(2) 求 v_i.

由

$$\frac{dv_i}{d\theta}=2\sqrt{-1}v_i^{(t)}v_i+r_ie_t-\sqrt{-1}e_t,$$

得

$$\begin{aligned}\frac{d^2v_i}{d\theta^2}&=2\sqrt{-1}v_i^{(t)}\frac{dv_i}{d\theta}+2\sqrt{-1}\frac{dv_i^{(t)}}{d\theta}v_i+\frac{dr_i}{d\theta}e_t\\&=2\sqrt{-1}v_i^{(t)}(2\sqrt{-1}v_i^{(t)}v_i+r_ie_t-\sqrt{-1}e_t)\\&\quad+2\sqrt{-1}(2\sqrt{-1}{v_i^{(t)}}^2+r_i-\sqrt{-1})v_i+(2\sqrt{-1}r_iv_i^{(t)}-2v_i^{(t)})e_t\\&=\ -8{v_i^{(t)}}^2v_i+4\sqrt{-1}v_i^{(t)}r_ie_t+2\sqrt{-1}(r_i-\sqrt{-1})v_i\\&=4\sqrt{-1}v_i^{(t)}(2\sqrt{-1}v_i^{(t)}v_i+r_ie_t-\sqrt{-1}e_t)+2\sqrt{-1}(r_i-\sqrt{-1})v_i-4v_i^{(t)}e_t\\&=4\sqrt{-1}v_i^{(t)}\frac{dv_i}{d\theta}+2\sqrt{-1}(r_i-\sqrt{-1})v_i-4v_i^{(t)}e_t.\end{aligned}$$

令 $f(\theta)=\sqrt{c-1}\sin(2\theta+c')+\sqrt{-1}$, 可以得到下面重要的关系式

$$\begin{aligned}&r_i+\sqrt{-1}=-\frac{2}{f(\theta)},\\&f''(\theta)=4\sqrt{-1}-4f(\theta),\\&v_i^{(t)}=\frac{\frac{dr_i}{d\theta}}{2\sqrt{-1}(r_i+\sqrt{-1})}=-\frac{f'(\theta)}{2\sqrt{-1}f(\theta)}.\end{aligned}$$

$$\frac{d^2v_i}{d\theta^2}=-2\frac{f'(\theta)}{f(\theta)}\frac{dv_i}{d\theta}-4\sqrt{-1}\frac{1}{f(\theta)}v_i+4v_i-2\sqrt{-1}\frac{f'(\theta)}{f(\theta)}e_t,$$

即

$$f(\theta)\frac{d^2v_i}{d\theta^2}=-2f'(\theta)\frac{dv_i}{d\theta}-f''(\theta)v_i-2\sqrt{-1}f'(\theta)e_t.$$

$$(f(\theta)v_i)''=-2\sqrt{-1}f'(\theta)e_t,$$

所以

$$v_i=\frac{\sqrt{-1}\sqrt{c-1}\cos(2\theta+c')e_t+(2e_t+c_1)\theta+c_2}{\sqrt{c-1}\sin(2\theta+c')+\sqrt{-1}}, \tag{3.3.3}$$

由 (3.3.3) 式及初值条件 $v_i(0)=u_i$ 可得

$$c_1=-2e_t,\quad c_2=\frac{-2u_i+2u_i^{(t)}e_t}{s_i+\sqrt{-1}}.$$

所以

$$v_i=\frac{\sqrt{-1}\sqrt{c-1}\cos(2\theta+c')e_t+c_2}{\sqrt{c-1}\sin(2\theta+c')+\sqrt{-1}},\quad c_2=\frac{-2u_i+2u_i^{(t)}e_t}{s_i+\sqrt{-1}}.$$

(3) $w_{pi}(p<i)$ 和 $w_{ip}(p>i)$ 的求法相同，这里只写出 $w_{pi}(p<i)$ 的求法. 由

$$\frac{dw_{pi}}{d\theta}=\sqrt{-1}\sum_s(r_i+\sqrt{-1})v_pQ_{pi}^{(s)}e_t'e_s+\sqrt{-1}\sum_{s,r}w_{pi}^{(s)}v_i\overline{(Q_{pi}^{(r)})'}Q_{pi}^{(s)}e_t'e_r,$$

得

$$\begin{aligned}\frac{d^2w_{pi}}{d\theta^2}&=\sqrt{-1}\sum_s\frac{dr_i}{d\theta}v_pQ_{pi}^{(s)}e_t'e_s+\sqrt{-1}\sum_s(r_i+\sqrt{-1})\frac{dv_p}{d\theta}Q_{pi}^{(s)}e_t'e_s\\&\quad+\sqrt{-1}\sum_{s,r}\frac{dw_{pi}^{(s)}}{d\theta}v_i\overline{(Q_{pi}^{(r)})'}Q_{pi}^{(s)}e_t'e_r+\sqrt{-1}\sum_{s,r}w_{pi}^{(s)}\frac{dv_i}{d\theta}\overline{(Q_{pi}^{(r)})'}Q_{pi}^{(s)}e_t'e_r\\&=-\sum_{s,r}(r_i+\sqrt{-1})v_pQ_{pi}^{(r)}e_t'v_i\overline{(Q_{pi}^{(r)})'}Q_{pi}^{(s)}e_t'e_s\\&\quad+\sqrt{-1}\sum_{s,r}(r_i+\sqrt{-1})w_{pi}^{(s)}e_t\overline{(Q_{pi}^{(s)})'}Q_{pi}^{(r)}e_t'e_r-2\sum_s(r_i+\sqrt{-1})v_i^{(t)}v_pQ_{pi}^{(s)}e_t'e_s\\&\quad-2\sum_{s,r}v_i^{(t)}w_{pi}^{(s)}v_i\overline{(Q_{pi}^{(r)})'}Q_{pi}^{(s)}e_t'e_r+\sqrt{-1}\sum_{s,r}w_{pi}^{(s)}(r_i-\sqrt{-1})e_t\overline{(Q_{pi}^{(r)})'}Q_{pi}^{(s)}e_t'e_r\\&\quad-\sum_{s,r}(r_i+\sqrt{-1})v_i\overline{(Q_{pi}^{(r)})'}Q_{pi}^{(s)}e_t'v_pQ_{pi}^{(s)}e_t'e_r-\sum_{s,r,l}w_{pi}^{(l)}v_i\overline{(Q_{pi}^{(r)})'}Q_{pi}^{(s)}e_t'v_i\overline{(Q_{pi}^{(s)})'}Q_{pi}^{(l)}e_t'e_r\end{aligned}$$

$$
\begin{aligned}
&= (r_i+\sqrt{-1})\sum_{s,r} v_p Q_{pi}^{(r)} e_t' v_i \overline{(Q_{pi}^{(s)})}' Q_{pi}^{(r)} e_t' e_s - 2(r_i+\sqrt{-1})\sum_{s} v_p Q_{pi}^{(s)} e_t' v_i e_t' e_s \\
&\quad - \sqrt{-1}\sum_{s,r}(r_i+\sqrt{-1}) w_{pi}^{(s)} e_t \overline{(Q_{pi}^{(r)})}' Q_{pi}^{(s)} e_t' e_r + 2\sqrt{-1}\sum_{s}(r_i+\sqrt{-1}) w_{pi}^{(s)} e_t e_t' e_s \\
&\quad - 2(r_i+\sqrt{-1})\sum_{s} v_i^{(t)} v_p Q_{pi}^{(s)} e_t' e_s - 2\sum_{s,r} v_i^{(t)} w_{pi}^{(s)} v_i \overline{(Q_{pi}^{(r)})}' Q_{pi}^{(s)} e_t' e_r \\
&\quad + \sqrt{-1}\sum_{s,r}(r_i-\sqrt{-1}) w_{pi}^{(s)} e_t \overline{(Q_{pi}^{(r)})}' Q_{pi}^{(s)} e_t' e_r - \sum_{s,r}(r_i+\sqrt{-1}) v_i \overline{(Q_{pi}^{(r)})}' Q_{pi}^{(s)} e_t' e_t Q_{pi}^{(s)'} v_p' e_r \\
&\quad + \sum_{s,r,l} w_{pi}^{(l)} v_i \overline{(Q_{pi}^{(r)})}' Q_{pi}^{(s)} e_t' v_i \overline{(Q_{pi}^{(l)})}' Q_{pi}^{(s)} e_t' e_r - 2\sum_{s,r} w_{pi}^{(s)} v_i \overline{(Q_{pi}^{(r)})}' Q_{pi}^{(s)} e_t' v_i e_t' e_r \\
&= -4(r_i+\sqrt{-1})\sum_{s} v_p Q_{pi}^{(s)} v_i^{(t)} e_t' e_s + 2\sqrt{-1}(r_i+\sqrt{-1}) w_{pi} \\
&\quad - 4\sum_{s,r} v_i^{(t)} w_{pi}^{(s)} v_i \overline{(Q_{pi}^{(r)})}' Q_{pi}^{(s)} e_t' e_r + 2\sum_{s,r} w_{pi}^{(s)} e_t \overline{(Q_{pi}^{(r)})}' Q_{pi}^{(s)} e_t' e_r \\
&= 4\sqrt{-1} v_i^{(t)} \frac{dw_{pi}}{d\theta} + 2\sqrt{-1}(r_i+\sqrt{-1}) w_{pi} + 2\sum_{s,r} w_{pi} e_s' e_t \overline{(Q_{pi}^{(r)})}' Q_{pi}^{(s)} e_t' e_r
\end{aligned}
$$

所以可列出满足初值条件的微分方程

$$
\begin{cases}
\frac{d^2 w_{pi}}{d\theta^2} = 4\sqrt{-1} v_i^{(t)} \frac{dw_{pi}}{d\theta} + 2\sqrt{-1}(r_i+\sqrt{-1}) w_{pi} + 2\sum_{s,r} w_{pi} e_s' e_t \overline{(Q_{pi}^{(r)})}' Q_{pi}^{(s)} e_t' e_r \\
w_{pi}(0) = z_{pi}, \\
\frac{dw_{pi}}{d\theta}(0) = \sqrt{-1}\sum_{s}(s_i+\sqrt{-1}) u_p Q_{pi}^{(s)} e_t' e_s + \sqrt{-1}\sum_{s,r} z_{pi}^{(s)} u_i \overline{(Q_{pi}^{(r)})}' Q_{pi}^{(s)} e_t' e_r
\end{cases}
$$

$$
\frac{d^2 w_{pi}}{d\theta^2} = 4\sqrt{-1}\left(-\frac{f'(\theta)}{2\sqrt{-1}f(\theta)}\right)\frac{dw_{pi}}{d\theta} + 2\sqrt{-1}\frac{-2}{f(\theta)} w_{pi} + 2w_{pi}\sum_{s,r} e_s' e_t \overline{(Q_{pi}^{(r)})}' Q_{pi}^{(s)} e_t' e_r,
$$

$$
f(\theta)\frac{d^2 w_{pi}}{d\theta^2} = -2f'(\theta)\frac{dw_{pi}}{d\theta} - 4\sqrt{-1} w_{pi} + 2f(\theta) w_{pi}\sum_{s,r} e_s' e_t \overline{(Q_{pi}^{(r)})}' Q_{pi}^{(s)} e_t' e_r,
$$

$$
\begin{aligned}
(f(\theta) w_{pi})'' &= -4f(\theta) w_{pi} + 2f(\theta) w_{pi}\sum_{s,r} e_s' e_t \overline{(Q_{pi}^{(r)})}' Q_{pi}^{(s)} e_t' e_r \\
&= -2f(\theta) w_{pi}\sum_{s,r} e_s' e_t \overline{(Q_{pi}^{(s)})}' Q_{pi}^{(r)} e_t' e_r,
\end{aligned}
$$

令

$$
x = f(\theta) w_{pi}, \quad K_{pi}^{(t)} = \sum_{r} Q_{pi}^{(r)} e_t' e_r,
$$

则

$$
x'' = -2x\overline{(K_{pi}^{(t)})}' K_{pi}^{(t)},
$$

$H=\overline{(K_{pi}^{(t)})}'K_{pi}^{(t)}$ 为正定的 Hermite 矩阵，所以，存在酉矩阵 U_{pi}, 使得 $\overline{U}_{pi}{}'HU_{pi}$ 是对角阵 $\{\lambda_{pi}^{(s)}\}$, $\lambda_{pi}^{(s)}\in\mathbb{R}^+$, $s=1,\cdots,n_{pi}$ 为矩阵 H 的特征值.

$$\frac{d^2x}{d\theta^2}=-2xH=-2xU_{pi}\begin{pmatrix}\lambda_{pi}^{(1)} & & \\ & \ddots & \\ & & \lambda_{pi}^{(n_{pi})}\end{pmatrix}\overline{U}_{pi}{}',$$

令 $y=xU_{pi}$, 则

$$\frac{d^2y}{d\theta^2}=-2y\begin{pmatrix}\lambda_{pi}^{(1)} & & \\ & \ddots & \\ & & \lambda_{pi}^{(n_{pi})}\end{pmatrix},$$

记： $y=(y_1,y_2,\cdots,y_{n_{pi}})$, 而 $\frac{d^2y_s}{d\theta^2}=-2\lambda_{pi}^{(s)}y_s$ 的解为

$$y_s=c_{pi}^{(s)}\sin(\sqrt{2\lambda_{pi}^{(s)}}\theta+c_{pi}^{(s)'}),$$

$$xU_{pi}=f(\theta)w_{pi}U_{pi}=y=\sum_s y_se_s=\sum_s c_{pi}^{(s)}\sin(\sqrt{2\lambda_{pi}^{(s)}}\theta+c_{pi}^{(s)'})e_s,$$

所以

$$w_{pi}=\frac{\sum\limits_s c_{pi}^{(s)}\sin(\sqrt{2\lambda_{pi}^{(s)}}\theta+c_{pi}^{(s)'})e_s\overline{U}_{pi}{}'}{f(\theta)}. \tag{3.3.4}$$

由 (3.3.4) 式及初值条件得

$$c_{pi}^{(s)'}=\operatorname{arctg}\sqrt{2\lambda_{pi}^{(s)}}\frac{x(0)U_{pi}e_s'}{x'(0)U_{pi}e_s'},\qquad c_{pi}^{(s)}=\frac{f(0)z_{pi}U_{pi}e_s'}{\sin c_{pi}^{(s)'}}.$$

所以

$$w_{pi}=\frac{1}{f(\theta)}\sum_s\frac{\sin(\sqrt{2\lambda_{pi}^{(s)}}\theta+c_{pi}^{(s)'})}{\sin c_{pi}^{(s)'}}f(0)z_{pi}U_{pi}e_s'e_s\overline{U}_{pi}{}',$$

$$c_{pi}^{(s)'}=\operatorname{arctg}\frac{\sqrt{2\lambda_{pi}^{(s)}}f(0)z_{pi}U_{pi}e_s'}{(\sqrt{-1}f(0)(\sum\limits_s(s_i+\sqrt{-1})u_pQ_{pi}^{(s)}e_t'e_s+\sum\limits_{s,r}z_{pi}^{(s)}u_i\overline{(Q_{pi}^{(r)})}'Q_{pi}^{(s)}e_t'e_r)+f'(0)z_{pi})U_{pi}e_s'}.$$

(4) $r_p(p<i)$ 和 $r_p(p>i)$ 的求法相同，下面只写出 $r_p(p<i)$ 的求法.
由

$$\frac{dr_p}{d\theta}=2\sqrt{-1}\sum_s w_{pi}^{(s)}v_pQ_{pi}^{(s)}e_t',$$

得

$$
\begin{aligned}
\frac{d^2r_p}{d\theta^2} &= 2\sqrt{-1}\sum_s \frac{dw_{pi}^{(s)}}{d\theta} v_p Q_{pi}^{(s)} e_t' + 2\sqrt{-1}\sum_s w_{pi}^{(s)} \frac{dv_p}{d\theta} Q_{pi}^{(s)} e_t' \\
&= -2\sum_s (r_i+\sqrt{-1}) v_p Q_{pi}^{(s)} e_t' v_p Q_{pi}^{(s)} e_t' - 2\sum_{s,l} w_{pi}^{(l)} v_i \overline{(Q_{pi}^{(s)})}' Q_{pi}^{(l)} e_t' v_p Q_{pi}^{(s)} e_t' \\
&\quad + 2\sqrt{-1}\sum_{s,l} w_{pi}^{(s)} w_{pi}^{(l)} e_t \overline{(Q_{pi}^{(l)})}' Q_{pi}^{(s)} e_t' - 2\sum_{s,l} w_{pi}^{(s)} v_p Q_{pi}^{(l)} e_t' v_i \overline{(Q_{pi}^{(l)})}' Q_{pi}^{(s)} e_t' \\
&= -8v_i^{(t)} \sum_s w_{pi}^{(s)} v_p Q_{pi}^{(s)} e_t' + 2\sqrt{-1} w_{pi} w_{pi}' \\
&= 4\sqrt{-1} v_i^{(t)} \frac{dr_p}{d\theta} + 2\sqrt{-1} w_{pi} w_{pi}' \\
&= 4\sqrt{-1}\left(-\frac{f'(\theta)}{2\sqrt{-1}f(\theta)}\right)\frac{dr_p}{d\theta} + 2\sqrt{-1} w_{pi} w_{pi}',
\end{aligned}
$$

所以

$$
f^2(\theta)\frac{d^2r_p}{d\theta^2} = -2f(\theta)f'(\theta)\frac{dr_p}{d\theta} + 2\sqrt{-1}f^2(\theta) w_{pi} w_{pi}',
$$

$$
\left(f^2(\theta)\frac{dr_p}{d\theta}\right)' = 2\sqrt{-1}f^2(\theta) w_{pi} w_{pi}',
$$

将已求得的 w_{pi} 代入得

$$
\begin{aligned}
r_p = \sum_{s,r} & \frac{2\sqrt{-1}f^2(0) z_{pi} U_{pi} E_{ss} \overline{U}_{pi}{}' \overline{U}_{pi} E_{rr} U_{pi}' z_{pi}'}{\sin c_{pi}^{(s)'} \sin c_{pi}^{(r)'}} \\
& \cdot \int \left(\frac{1}{f^2(\theta)} \int \sin\left(\sqrt{2\lambda_{pi}^{(s)}}\theta + c_{pi}^{(s)'}\right) \sin\left(\sqrt{2\lambda_{pi}^{(r)}}\theta + c_{pi}^{(r)'}\right) d\theta\right) d\theta,
\end{aligned}
$$

待定参数由初值 $r_p(0)=s_p$，$\frac{dr_p}{d\theta}(0) = 2\sqrt{-1}\sum_s z_{pi}^{(s)} u_p Q_{pi}^{(s)} e_t'$ 确定.

(5) $v_p(p<i)$ 和 $v_p(p>i)$ 的求法相同，下面只写出 $v_p(p<i)$ 的求法.

$$
\frac{dv_p}{d\theta} = \sum_s \left[w_{pi}^{(s)} e_t \overline{(Q_{pi}^{(s)})}' + \sqrt{-1} v_p Q_{pi}^{(s)} e_t' v_i \overline{(Q_{pi}^{(s)})}'\right], \quad p<i,
$$

由 $\frac{dw_{pi}^{(s)}}{d\theta} = \sqrt{-1}(r_i+\sqrt{-1}) v_p Q_{pi}^{(s)} e_t' + \sqrt{-1}\sum_l w_{pi}^{(l)} v_i \overline{(Q_{pi}^{(s)})}' Q_{pi}^{(l)} e_t', \quad r_i+\sqrt{-1} = -\frac{2}{f(\theta)}$，得

$$
\sqrt{-1} v_p Q_{pi}^{(s)} e_t' = \frac{\sqrt{-1}}{2} f(\theta) \sum_l w_{pi}^{(l)} v_i \overline{(Q_{pi}^{(s)})}' Q_{pi}^{(l)} e_t' - \frac{1}{2} f(\theta) \frac{dw_{pi}^{(s)}}{d\theta},
$$

$$\frac{dv_p}{d\theta}=\sum_s[w_{pi}^{(s)}e_t\overline{(Q_{pi}^{(s)})}'+\frac{\sqrt{-1}}{2}f(\theta)\sum_l w_{pi}^{(l)}v_i\overline{(Q_{pi}^{(s)})}'Q_{pi}^{(l)}e_t'v_i\overline{(Q_{pi}^{(s)})}'-\frac{1}{2}f(\theta)\frac{dw_{pi}^{(s)}}{d\theta}v_i\overline{(Q_{pi}^{(s)})}']$$

$$=\sum_s[w_{pi}^{(s)}e_t\overline{(Q_{pi}^{(s)})}'-\frac{1}{2}f(\theta)\frac{dw_{pi}^{(s)}}{d\theta}v_i\overline{(Q_{pi}^{(s)})}'],$$

将以求得的 $w_{pi}^{(s)}, v_i$ 代入得

$$v_p=\sum_{s,r}f(0)z_{pi}U_{pi}E_{rr}\overline{U}_{pi}'E_{st}\overline{(Q_{pi}^{(s)})}'\int\frac{\sin(\sqrt{2\lambda_{pi}^{(r)}}\theta+c_{pi}^{(r)'})}{\sin c_{pi}^{(r)'}f(\theta)}d\theta-\frac{1}{2}\sum_{s,r}f(0)z_{pi}U_{pi}E_{rr}\overline{U}_{pi}'e_s'$$

$$\cdot(\int\frac{\sqrt{2\lambda_{pi}^{(r)}}\cos(\sqrt{2\lambda_{pi}^{(r)}}\theta+c_{pi}^{(r)'})f(\theta)-\sin(\sqrt{2\lambda_{pi}^{(r)}}\theta+c_{pi}^{(r)'})f'(\theta)}{\sin c_{pi}^{(r)'}f^2(\theta)}$$

$$(\sqrt{-1}\sqrt{c-1}\cos(2\theta+c')e_t+c_2)d\theta)\overline{(Q_{pi}^{(s)})}',$$

待定参数由初值 $v_p(0)=u_p$ 确定.

(6) 求 $w_{lp}(l<p<i)$, $w_{lp}(l<i<p)$ 和 $w_{lp}(l<p<i)$ 求法相同，下面只写出 $w_{lp}(l<p<i)$ 的求法.

由

$$\frac{dw_{lp}}{d\theta}=\sqrt{-1}\sum_{s,u}[v_pQ_{pi}^{(s)}e_t'w_{li}A_{lp}^{ui}e_s'e_u+v_lQ_{li}^{(s)}e_t'e_sA_{lp}^{ui}w_{pi}'e_u],$$

得

$$\frac{d^2w_{lp}}{d\theta^2}=\sqrt{-1}\sum_{s,u}\frac{dv_p}{d\theta}Q_{pi}^{(s)}e_t'w_{li}A_{lp}^{ui}e_s'e_u+\sqrt{-1}\sum_{s,u}v_pQ_{pi}^{(s)}e_t'\frac{dw_{li}}{d\theta}A_{lp}^{ui}e_s'e_u$$

$$+\sqrt{-1}\sum_{s,u}\frac{dv_l}{d\theta}Q_{li}^{(s)}e_t'e_sA_{lp}^{ui}w_{pi}'e_u+\sqrt{-1}\sum_{s,u}v_lQ_{li}^{(s)}e_t'e_sA_{lp}^{ui}\frac{dw_{pi}'}{d\theta}e_u$$

$$=-\sum_{s,r,u}v_pQ_{pi}^{(r)}e_t'v_i\overline{(Q_{pi}^{(r)})}'Q_{pi}^{(s)}e_t'w_{li}A_{lp}^{ui}e_s'e_u+\sqrt{-1}\sum_{s,r,u}w_{pi}^{(r)}e_t\overline{(Q_{pi}^{(r)})}'Q_{pi}^{(s)}e_t'w_{li}A_{lp}^{ui}e_s'e_u$$

$$-(r_i+\sqrt{-1})\sum_{s,r,u}v_pQ_{pi}^{(s)}e_t'v_lQ_{li}^{(r)}e_t'e_rA_{lp}^{ui}e_s'e_u-\sum_{s,r,u,h}w_{li}^{(h)}v_pQ_{pi}^{(s)}e_t'v_i\overline{(Q_{li}^{(r)})}'Q_{li}^{(h)}e_t'e_rA_{lp}^{ui}e_s'e_u$$

$$-\sum_{s,r,u}v_lQ_{li}^{(r)}e_t'v_i\overline{(Q_{li}^{(r)})}'Q_{li}^{(s)}e_t'e_sA_{lp}^{ui}w_{pi}'e_u+\sqrt{-1}\sum_{s,r,u}w_{li}^{(r)}e_t\overline{(Q_{li}^{(r)})}'Q_{li}^{(s)}e_t'e_sA_{lp}^{ui}w_{pi}'e_u$$

$$-(r_i+\sqrt{-1})\sum_{s,r,u}v_lQ_{li}^{(s)}e_t'e_sA_{lp}^{ui}e_r'v_pQ_{pi}^{(r)}e_t'e_u-\sum_{s,r,u,h}w_{pi}^{(h)}v_lQ_{li}^{(s)}e_t'e_sA_{lp}^{ui}e_r'v_i\overline{(Q_{pi}^{(r)})}'Q_{pi}^{(h)}e_t'e_u$$

$$
\begin{aligned}
=& -2v_i^{(t)}\sum_{s,u} v_p Q_{pi}^{(s)} e_t' w_{li} A_{lp}^{ui} e_s' e_u + \sqrt{-1}\sum_{s,r,u} w_{pi}^{(r)} e_t \overline{(Q_{pi}^{(r)})}' Q_{pi}^{(s)} e_t' w_{li} A_{lp}^{ui} e_s' e_u \\
& - \sum_{s,r,u} v_p Q_{pi}^{(s)} e_t' v_i \overline{(Q_{pi}^{(s)})}' Q_{pi}^{(r)} e_t' w_{li} A_{lp}^{ui} e_r' e_u - 2v_i^{(t)} \sum_{s,u} v_l Q_{li}^{(s)} e_t' e_s A_{lp}^{ui} w_{pi}' e_u \\
& + \sqrt{-1}\sum_{s,r,u,h} w_{li}^{(r)} w_{pi}^{(h)} e_t \overline{(Q_{pi}^{(s)})}' Q_{pi}^{(h)} e_t' e_r A_{lp}^{ui} e_s' e_u - \sum_{s,r,u} w_{pi}^{(s)} v_l Q_{lp}^{(u)} Q_{pi}^{(r)} e_t' v_i \overline{(Q_{pi}^{(r)})}' Q_{pi}^{(s)} e_t' e_u \\
=& -4v_i^{(t)} \sum_{s,u} (v_p Q_{pi}^{(s)} e_t' w_{li} A_{lp}^{ui} e_s' e_u + v_l Q_{li}^{(s)} e_t' e_s A_{lp}^{ui} w_{pi}' e_u) + 2\sqrt{-1}\sum_{s,u} w_{pi}^{(s)} w_{li} A_{lp}^{ui} e_s' e_u \\
=& 4\sqrt{-1} v_i^{(t)} \frac{dw_{lp}}{d\theta} + 2\sqrt{-1}\sum_{s,u} w_{pi}^{(s)} w_{li} A_{lp}^{ui} e_s' e_u
\end{aligned}
$$

以上计算利用了公式

$$
\sum_r e_r A_{lp}^{ui} e_s' Q_{li}^{(r)} = Q_{lp}^{(u)} Q_{pi}^{(s)},
$$

$$
\sum_s e_h A_{lp}^{ui} e_s' Q_{pi}^{(s)} = \overline{(Q_{lp}^{(u)})}' Q_{li}^{(h)}.
$$

所以

$$
\frac{d^2 w_{lp}}{d\theta^2} = -2\frac{f'(\theta)}{f(\theta)}\frac{dw_{lp}}{d\theta} + 2\sqrt{-1}\sum_{s,u} w_{pi}^{(s)} w_{li} A_{lp}^{ui} e_s' e_u,
$$

$$
f^2(\theta)\frac{d^2 w_{lp}}{d\theta^2} + 2f(\theta)f'(\theta)\frac{dw_{lp}}{d\theta} = 2\sqrt{-1} f^2(\theta)\sum_{s,u} w_{pi}^{(s)} w_{li} A_{lp}^{ui} e_s' e_u,
$$

$$
(f^2(\theta)\frac{dw_{lp}}{d\theta})' = 2\sqrt{-1} f^2(\theta)\sum_{s,u} w_{pi}^{(s)} w_{li} A_{lp}^{ui} e_s' e_u,
$$

将已求得的 $w_{pi}^{(s)}$, w_{li} 代入得

$$
\begin{aligned}
w_{lp} = & \sum_{s,r,u,h} \frac{2\sqrt{-1} f^2(0) z_{pi} U_{pi} E_{rr} \overline{U}_{pi}' e_s' z_{li} U_{li} E_{hh} \overline{U}_{li}'}{\sin c_{pi}^{(r)'} \sin c_{li}^{(h)'}} \\
& \int (\frac{1}{f^2(\theta)} \int \sin(\sqrt{2\lambda_{pi}^{(r)}}\theta + c_{pi}^{(r)'}) \sin(\sqrt{2\lambda_{li}^{(h)}}\theta + c_{li}^{(h)'}) A_{lp}^{ui} e_s' e_u d\theta) d\theta,
\end{aligned}
$$

待定参数由初值条件

$$
w_{lp}(0) = z_{lp}, \quad \frac{dw_{lp}}{d\theta}(0) = \sqrt{-1}\sum_{s,u}[u_p Q_{pi}^{(s)} e_t' z_{li} A_{lp}^{ui} e_s' e_u + u_l Q_{li}^{(s)} e_t' e_s A_{lp}^{ui} z_{pi}' e_u]
$$

确定.

综上， $\exp(\theta(Y_i^{(t)} - \widetilde{P}_i^{(t)}))$ 可表示为：

$$r_i = -\frac{2}{f(\theta)} - \sqrt{-1},$$

$$v_i = \frac{\sqrt{-1}\sqrt{c-1}\cos(2\theta + c')e_t + c_2}{f(\theta)},$$

$$w_{pi} = \frac{1}{f(\theta)} \sum_s \frac{\sin(\sqrt{2\lambda_{pi}^{(s)}}\theta + c_{pi}^{(s)'})}{\sin c_{pi}^{(s)'}} f(0) z_{pi} U_{pi} e_s' e_s \overline{U}_{pi}', \quad p < i,$$

$$w_{ip} = \frac{1}{f(\theta)} \sum_r \frac{\sin(\sqrt{2\mu_{ip}^{(r)}}\theta + c_{ip}^{(r)'})}{\sin c_{ip}^{(r)'}} f(0) z_{ip} U_{ip} e_r' e_r \overline{U}_{ip}', \quad p > i,$$

$$r_p = \sum_{s,r} \frac{2\sqrt{-1} f^2(0) z_{pi} U_{pi} E_{ss} \overline{U}_{pi}' \overline{U}_{pi} E_{rr} U_{pi}' z_{pi}'}{\sin c_{pi}^{(s)'} \sin c_{pi}^{(r)'}}$$

$$\cdot \int (\frac{1}{f^2(\theta)} \int \sin(\sqrt{2\lambda_{pi}^{(s)}}\theta + c_{pi}^{(s)'}) \sin(\sqrt{2\lambda_{pi}^{(r)}}\theta + c_{pi}^{(r)'}) d\theta) d\theta, \quad p < i,$$

$$r_p = \sum_{s,r} \frac{2\sqrt{-1} f^2(0) z_{ip} U_{ip} E_{ss} \overline{U}_{ip}' \overline{U}_{ip} E_{rr} U_{ip}' z_{ip}'}{\sin c_{ip}^{(s)'} \sin c_{ip}^{(r)'}}$$

$$\cdot \int (\frac{1}{f^2(\theta)} \int \sin(\sqrt{2\mu_{ip}^{(s)}}\theta + c_{ip}^{(s)'}) \sin(\sqrt{2\mu_{ip}^{(r)}}\theta + c_{ip}^{(r)'}) d\theta) d\theta, \quad p > i,$$

$$v_p = \sum_{s,r} f(0) z_{pi} U_{pi} E_{rr} \overline{U}_{pi}' E_{st} \overline{(Q_{pi}^{(s)})}' \int \frac{\sin(\sqrt{2\lambda_{pi}^{(r)}}\theta + c_{pi}^{(r)'})}{\sin c_{pi}^{(r)'} f(\theta)} d\theta - \frac{1}{2} \sum_{s,r} f(0) z_{pi} U_{pi} E_{rr} \overline{U}_{pi}' e_s'$$

$$\cdot (\int \frac{\sqrt{2\lambda_{pi}^{(r)}} \cos(\sqrt{2\lambda_{pi}^{(r)}}\theta + c_{pi}^{(r)'}) f(\theta) - \sin(\sqrt{2\lambda_{pi}^{(r)}}\theta + c_{pi}^{(r)'}) f'(\theta)}{\sin c_{pi}^{(r)'} f^2(\theta)}$$

$$(\sqrt{-1}\sqrt{c-1}\cos(2\theta + c')e_t + c_2) d\theta) \overline{(Q_{pi}^{(s)})}', \quad p < i,$$

$$v_p = \sum_{s,r} f(0) z_{ip} U_{ip} E_{rr} \overline{U}_{ip}' E_{st} Q_{ip}^{(s)} \int \frac{\sin(\sqrt{2\mu_{ip}^{(r)}}\theta + c_{ip}^{(r)'})}{\sin c_{ip}^{(r)'} f(\theta)} d\theta - \frac{1}{2} \sum_{s,r} f(0) z_{ip} U_{ip} E_{rr} \overline{U}_{ip}' e_s'$$

$$\cdot (\int \frac{\sqrt{2\mu_{ip}^{(r)}} \cos(\sqrt{2\mu_{ip}^{(r)}}\theta + c_{ip}^{(r)'}) f(\theta) - \sin(\sqrt{2\mu_{ip}^{(r)}}\theta + c_{ip}^{(r)'}) f'(\theta)}{\sin c_{ip}^{(r)'} f^2(\theta)}$$

$$(\sqrt{-1}\sqrt{c-1}\cos(2\theta + c')e_t + c_2) d\theta) Q_{ip}^{(s)}, \quad p > i,$$

$$w_{lp} = \sum_{s,r,u,h} \frac{2\sqrt{-1} f^2(0) z_{pi} U_{pi} E_{rr} \overline{U}_{pi}' e_s' z_{li} U_{li} E_{hh} \overline{U}_{li}' A_{lp}^{ui} e_s' e_u}{\sin c_{pi}^{(r)'} \sin c_{li}^{(h)'}}$$

$$\cdot \int (\frac{1}{f^2(\theta)} \int \sin(\sqrt{2\lambda_{pi}^{(r)}}\theta + c_{pi}^{(r)'}) \sin(\sqrt{2\lambda_{li}^{(h)}}\theta + c_{li}^{(h)'}) d\theta) d\theta, \quad l < p < i,$$

$$w_{lp}=\sum_{s,r,u}\frac{2\sqrt{-1}f^2(0)z_{li}U_{li}E_{rr}\overline{U}_{li}{}'e_s'z_{ip}U_{ip}E_{uu}\overline{U}_{ip}{}'(A_{li}^{sp})'}{\sin c_{li^{(r)}}'\sin c_{ip}^{(u)'}}$$

$$\cdot\int(\frac{1}{f^2(\theta)}\int\sin(\sqrt{2\lambda_{li}^{(r)}}\theta+c_{li}^{(r)'})\sin(\sqrt{2\mu_{ip}^{(u)}}\theta+c_{ip}^{(u)'})d\theta)d\theta,\quad l<i<p.$$

$$w_{lp}=\sum_{s,r,u}\frac{2\sqrt{-1}f^2(0)z_{il}U_{il}E_{rr}\overline{U}_{il}{}'e_s'z_{ip}U_{ip}E_{uu}\overline{U}_{ip}{}'A_{il}^{sp}}{\sin c_{il}^{(r)'}\sin c_{ip}^{(u)'}}$$

$$\cdot\int(\frac{1}{f^2(\theta)}\int\sin(\sqrt{2\mu_{il}^{(r)}}\theta+c_{il}^{(r)'})\sin(\sqrt{2\mu_{ip}^{(u)}}\theta+c_{ip}^{(u)'})d\theta)d\theta,\quad i<l<p.$$

其中

$$c=\frac{4\sqrt{-1}s_i-4u_i^{(t)2}}{(s_i+\sqrt{-1})^2},\quad c'=\operatorname{arctg}\frac{\sqrt{-1}-s_i}{2u_i^{(t)}},\quad f(\theta)=\sqrt{c-1}\sin(2\theta+c')+\sqrt{-1};$$

在 v_i 的表达式中， $c_2=\frac{-2u_i+2u_i^{(t)}e_t}{s_i+\sqrt{-1}}$;

在 w_{pi} 的表达式中，酉矩阵 U_{pi} 使 Hermite 矩阵 $H=\sum\limits_{s,r}e_s'e_t\overline{(Q_{pi}^{(s)})}'Q_{pi}^{(r)}e_t'e_r$ 对角化， $\lambda_{pi}^{(s)}$, $s=1,\cdots,n_{pi}$ 为 H 的特征值，

$$c_{pi}^{(s)'}=\operatorname{arctg}\frac{\sqrt{2\lambda_{pi}^{(s)}}f(0)z_{pi}U_{pi}e_s'}{(\sqrt{-1}f(0)(\sum\limits_s(s_i+\sqrt{-1})u_pQ_{pi}^{(s)}e_t'e_s+\sum\limits_{s,r}z_{pi}^{(s)}u_i\overline{(Q_{pi}^{(r)})}'Q_{pi}^{(s)}e_t'e_r)+f'(0)z_{pi})U_{pi}e_s'};$$

在 w_{ip} 的表达式中，酉矩阵 U_{ip} 使 Hermite 矩阵 $M=\sum\limits_{s,r}e_s'e_tQ_{ip}^{(s)}\overline{(Q_{ip}^{(r)})}'e_t'e_r$ 对角化， $\mu_{ip}^{(r)},r=1,\cdots,n_{ip}$ 为 M 的特征值，

$$c_{ip}^{(r)'}=\operatorname{arctg}\frac{\sqrt{2\mu_{ip}^{(r)}}f(0)z_{ip}U_{ip}e_r'}{(\sqrt{-1}f(0)(\sum\limits_s(s_i+\sqrt{-1})u_p\overline{(Q_{ip}^{(s)})}'e_t'e_s+\sum\limits_{s,r}z_{ip}^{(s)}u_iQ_{ip}^{(r)}\overline{(Q_{ip}^{(s)})}'e_t'e_r)+f'(0)z_{ip})U_{ip}e_r'};$$

在 $r_p(p<i)$ 的表达式中,待定参数由初值 $r_p(0)=s_p$, $\frac{dr_p}{d\theta}(0)=2\sqrt{-1}\sum\limits_s z_{pi}^{(s)}u_pQ_{pi}^{(s)}e_t'$ 确定;

在 $r_p(p>i)$ 的表达式中,待定参数由初值 $r_p(0)=s_p$, $\frac{dr_p}{d\theta}(0)=2\sqrt{-1}\sum\limits_s z_{ip}^{(s)}u_p\overline{(Q_{ip}^{(s)})}'e_t'$ 确定;

在 $v_p(p<i)$ 的表达式中，待定参数由初值 $v_p(0)=u_p$ 确定;

在 $v_p(p<i)$ 的表达式中，待定参数由初值 $v_p(0)=u_p$ 确定;

在 $w_{lp}(l<p<i)$ 的表达式中，待定参数由初值 $w_{lp}(0)=z_{lp}$, $\frac{dw_{lp}}{d\theta}(0)=\sqrt{-1}\sum\limits_{s,u}[u_pQ_{pi}^{(s)}e_t'z_{li}A_{lp}^{ui}e_s'e_u+u_lQ_{li}^{(s)}e_t'e_sA_{lp}^{ui}z_{pi}'e_u]$ 确定;

在 $w_{lp}(i<l<p)$ 的表达式中，待定参数由初值 $w_{lp}(0)=z_{lp}$，
$\frac{dw_{lp}}{d\theta}(0)=\sqrt{-1}\sum_s u_l Q_{li}^{(s)} e_t' z_{ip}(A_{li}^{sp})' + \sqrt{-1}\sum_{s,u} z_{li}^{(u)} u_p \overline{(Q_{ip}^{(s)})}' e_t' e_s (A_{li}^{up})'$ 确定；

在 $w_{lp}(i<l<p)$ 的表达式中，待定参数由初值 $w_{lp}(0)=z_{lp}$，
$\frac{dw_{lp}}{d\theta}(0)=\sqrt{-1}\sum_s u_l \overline{(Q_{il}^{(s)})}' e_t' z_{ip} A_{il}^{sp} + \sqrt{-1}\sum_{s,u} z_{il}^{(u)} u_p \overline{(Q_{ip}^{(s)})}' e_t' e_s A_{il}^{up}$ 确定.

二 计算 $\widetilde{Y}_i^{(t)}+P_i^{(t)},\quad i\in\{i_1,\cdots,i_\rho\}$ **所决定的单参数子群** $\exp(\theta(\widetilde{Y}_i^{(t)}+P_i^{(t)}))$. **由引理** 2.5 **列出满足零点初值为** (z,u) **的常微分方程组**

$$
\begin{aligned}
\frac{dr_i}{d\theta} &= 2r_i v_i^{(t)} + 2\sqrt{-1}v_i^{(t)},\\
\frac{dr_p}{d\theta} &= 2\sum_s w_{pi}^{(s)} v_p Q_{pi}^{(s)} e_t', \quad p<i,\\
\frac{dr_p}{d\theta} &= 2\sum_s w_{ip}^{(s)} v_p \overline{(Q_{ip}^{(s)})}' e_t', \quad p>i,\\
\frac{dv_i}{d\theta} &= 2v_i^{(t)} v_i + \sqrt{-1}r_i e_t + e_t,\\
\frac{dv_p}{d\theta} &= \sum_s [\sqrt{-1}w_{pi}^{(s)} e_t \overline{(Q_{pi}^{(s)})}' + v_p Q_{pi}^{(s)} e_t' v_i \overline{(Q_{pi}^{(s)})}'], \quad p<i,\\
\frac{dv_p}{d\theta} &= \sum_s [\sqrt{-1}w_{ip}^{(s)} e_t Q_{ip}^{(s)} + v_p \overline{(Q_{ip}^{(s)})}' e_t' v_i Q_{ip}^{(s)}], \quad p>i,\\
\frac{dw_{pi}}{d\theta} &= \sum_s (r_i+\sqrt{-1}) v_p Q_{pi}^{(s)} e_t' e_s + \sum_{s,r} w_{pi}^{(s)} v_i \overline{(Q_{pi}^{(r)})}' Q_{pi}^{(s)} e_t' e_r, \quad p<i,\\
\frac{dw_{ip}}{d\theta} &= \sum_s (r_i+\sqrt{-1}) v_p \overline{(Q_{ip}^{(s)})}' e_t' e_s + \sum_{s,r} w_{ip}^{(s)} v_i Q_{ip}^{(r)} \overline{(Q_{ip}^{(s)})}' e_t' e_r, \quad p>i,\\
\frac{dw_{lp}}{d\theta} &= \sum_{s,u} [v_p Q_{pi}^{(s)} e_t' w_{li} A_{lp}^{ui} e_s' e_u + v_l Q_{li}^{(s)} e_t' e_s A_{lp}^{ui} w_{pi}' e_u], \quad l<p<i,\\
\frac{dw_{lp}}{d\theta} &= \sum_s v_l Q_{li}^{(s)} e_t' w_{ip} (A_{li}^{sp})' + \sum_{s,u} w_{li}^{(u)} v_p \overline{(Q_{ip}^{(s)})}' e_t' e_s (A_{li}^{up})', \quad l<i<p,\\
\frac{dw_{lp}}{d\theta} &= \sum_s v_l \overline{(Q_{il}^{(s)})}' e_t' w_{ip} A_{il}^{sp} + \sum_{s,u} w_{il}^{(u)} v_p \overline{(Q_{ip}^{(s)})}' e_t' e_s A_{il}^{up}, \quad i<l<p.
\end{aligned}
$$

同一中微分方程组的求解类似，计算结果即 $\exp(\theta(\widetilde{Y}_i^{(t)}+P_i^{(t)}))$ **的表达式见定理中** (3-ii).

§3.4 $\exp(t(B_i+\frac{\partial}{\partial s_i}))$, $\exp(\theta(T_{ij}^{(t)}+2\frac{\partial}{\partial z_{ij}^{(t)}}))$

在这一节，我们计算 $B_i+\frac{\partial}{\partial s_i}$, $T_{ij}^{(t)}+2\frac{\partial}{\partial z_{ij}^{(t)}}$ **所决定的单参数子群.**

一 计算 $B_i+\frac{\partial}{\partial s_i},\quad i\in\{i_1,\cdots,i_\rho\}$ **所决定的单参数子群** $\exp(t(B_i+\frac{\partial}{\partial s_i}))$.

由引理 2.5 列出满足零点初值为 (z,u) 的常微分方程组

$$
\begin{aligned}
&\frac{dr_p}{dt}=w_{pi}w'_{pi}, \quad p<i,\\
&\frac{dr_p}{dt}=w_{ip}w'_{ip}, \quad p>i,\\
&\frac{dr_i}{dt}=r_i^2+1,\\
&\frac{dw_{pi}}{dt}=r_iw_{pi}, \quad p<i,\\
&\frac{dw_{ip}}{dt}=r_iw_{ip}, \quad p>i,\\
&\frac{dw_{kp}}{dt}=\sum_s w_{ki}A_{kp}^{si}w'_{pi}e_s, \quad k<p<i,\\
&\frac{dw_{kp}}{dt}=\sum_s w_{ki}^{(s)}w_{ip}(A_{ki}^{sp})', \quad k<i<p,\\
&\frac{dw_{kp}}{dt}=\sum_s w_{ik}^{(s)}w_{ip}A_{ik}^{sp}, \quad i<k<p,\\
&\frac{dv_p}{dt}=\sum_s w_{pi}^{(s)}v_i\overline{(Q_{pi}^{(s)})}', \quad p<i,\\
&\frac{dv_i}{dt}=r_iv_i,\\
&\frac{dv_p}{dt}=\sum_s w_{ip}^{(s)}v_iQ_{ip}^{(s)}, \quad p>i.
\end{aligned}
$$

下面求解这个常微分方程组.

(1) 求 r_i.

$$
由 \quad \begin{cases} \frac{dr_i}{dt}=r_i^2+1 \\ r_i(0)=s_i \end{cases} \quad 得 \quad r_i=tg(t+\operatorname{arctg}s_i).
$$

(2) v_i, $w_{pi}(p<i)$, $w_{ip}(p>i)$ 的求法相同，这里只写出 v_i 的求法.

$$
由 \quad \begin{cases} \frac{dv_i}{dt}=r_iv_i=tg(t+\operatorname{arctg}s_i)v_i \\ v_i(0)=u_i \end{cases} \quad 得 \quad v_i=sec(t+\operatorname{arctg}s_i)\frac{u_i}{\sqrt{1+s_i^2}}.
$$

(3) $r_p(p>i)$, $r_p(p<i)$ 的求法相同，这里只写出 $r_p(p>i)$ 的求法.

$$
由 \quad \begin{cases} \frac{dr_p}{dt}=w_{ip}w'_{ip}=sec^2(t+\operatorname{arctg}s_i)\frac{1}{1+s_i^2}z_{ip}z'_{ip} \\ r_p(0)=s_p \end{cases} \quad 得
$$

$$
r_p=s_p+\frac{tg(t+\operatorname{arctg}s_i)-s_i}{1+s_i^2}z_{ip}z'_{ip}.
$$

(4) $w_{kp}(k<p<i)$, $w_{kp}(k<i<p)$, $w_{kp}(i<k<p)$ 的求法相同，这里只写出 $w_{kp}(k<p<i)$ 的求法.

由 $\begin{cases} \frac{dw_{kp}}{dt} = \sum\limits_s w_{ki} A^{si}_{kp} w'_{pi} e_s = \sum\limits_s sec^2(t+\operatorname{arctg} s_i)\frac{1}{1+s_i^2} z_{ki} A^{si}_{kp} z'_{pi} e_s \\ w_{kp}(0) = z_{kp} \end{cases}$ 得;

$$w_{kp} = z_{kp} + \sum_s \frac{tg(t+\operatorname{arctg} s_i) - s_i}{1+s_i^2} z_{ki} A^{si}_{kp} z'_{pi} e_s.$$

(5) $v_p(p<i)$, $v_p(p>i)$ 的求法相同，这里只写出 $v_p(p<i)$ 的求法.

由 $\begin{cases} \frac{dv_p}{dt} = \sum\limits_s w^{(s)}_{pi} v_i \overline{(Q^{(s)}_{pi})}' = \sum\limits_s sec^2(t+\operatorname{arctg} s_i)\frac{1}{1+s_i^2} z^{(s)}_{pi} u_i \overline{(Q^{(s)}_{pi})}' \\ v_p(0) = u_p \end{cases}$,得

$$v_p = u_p + \sum_s \frac{tg(t+\operatorname{arctg} s_i) - s_i}{1+s_i^2} z^{(s)}_{pi} u_i \overline{(Q^{(s)}_{pi})}',$$

引入新的参数 $\theta = tgt$, 则 $t=0$ 等价于 $\theta = 0$.

$$tg(t+\operatorname{arctg} s_i) = \frac{\theta + s_i}{1 - s_i\theta}, \quad sec(t+\operatorname{arctg} s_i) = \frac{\sqrt{(1+s_i^2)(1+\theta^2)}}{1-s_i\theta}.$$

所以, $\exp(\operatorname{arctg}\theta(B_i + \frac{\partial}{\partial s_i}))$ 可以表示为:

$$\begin{aligned}
r_i &= \frac{s_i+\theta}{1-s_i\theta}, \\
r_p &= s_p + \frac{\theta}{1-s_i\theta} z_{pi} z'_{pi}, \quad p<i, \\
r_p &= s_p + \frac{\theta}{1-s_i\theta} z_{ip} z'_{ip}, \quad p>i, \\
w_{pi} &= \frac{\sqrt{1+\theta^2}}{1-s_i\theta} z_{pi}, \quad p<i, \\
w_{ip} &= \frac{\sqrt{1+\theta^2}}{1-s_i\theta} z_{ip}, \quad p>i, \\
w_{kp} &= z_{kp} + \sum_s \frac{\theta}{1-s_i\theta} z_{ki} A^{si}_{kp} z'_{pi} e_s, \quad k<p<i, \\
w_{kp} &= z_{kp} + \sum_s \frac{\theta}{1-s_i\theta} z^{(s)}_{ki} z_{ip} (A^{sp}_{ki})', \quad k<i<p, \\
w_{kp} &= z_{kp} + \sum_s \frac{\theta}{1-s_i\theta} z^{(s)}_{ik} z_{ip} A^{sp}_{ik}, \quad i<k<p,
\end{aligned}$$

$$v_i = \frac{\sqrt{1+\theta^2}}{1-s_i\theta}u_i,$$
$$v_p = u_p + \sum_s \frac{\theta}{1-s_i\theta} z_{pi}^{(s)} u_i \overline{(Q_{pi}^{(s)})}', \quad p < i,$$
$$v_p = u_p + \sum_s \frac{\theta}{1-s_i\theta} z_{ip}^{(s)} u_i Q_{ip}^{(s)}, \quad p > i.$$

二　计算 $T_{ij}^{(t)} + 2\frac{\partial}{\partial z_{ij}^{(t)}}$, $1 \le t \le n_{ij}$, $i < j$, $i, j \in \{ i_1, \cdots, i_\rho \}$ **所确定的单参数子群** $\exp(\theta(T_{ij}^{(t)} + 2\frac{\partial}{\partial z_{ij}^{(t)}}))$.

由引理 2.5 列出满足零点初值为 (z, u) 的常微分方程组

$$\frac{dr_i}{d\theta} = 2r_i w_{ij}^{(t)},$$
$$\frac{dr_j}{d\theta} = 2r_j w_{ij}^{(t)},$$
$$\frac{dr_p}{d\theta} = 2\sum_s w_{pi}^{(s)} w_{pj} A_{pi}^{sj} e_t', \quad p < i,$$
$$\frac{dr_p}{d\theta} = 2\sum_s w_{ip}^{(s)} e_t A_{ip}^{sj} w_{pj}', \quad i < p < j,$$
$$\frac{dr_p}{d\theta} = 2w_{ip} A_{ij}^{tp} w_{jp}', \quad p > j,$$
$$\frac{dv_i}{d\theta} = r_i v_j \overline{(Q_{ij}^{(t)})}' + \sum_s w_{ij}^{(s)} v_i Q_{ij}^{(t)} \overline{(Q_{ij}^{(s)})}',$$
$$\frac{dv_j}{d\theta} = r_j v_i Q_{ij}^{(t)} + \sum_s w_{ij}^{(s)} v_j \overline{(Q_{ij}^{(t)})}' Q_{ij}^{(s)},$$
$$\frac{dv_p}{d\theta} = \sum_s [w_{pj} A_{pi}^{sj} e_t' v_i \overline{(Q_{pi}^{(s)})}' + w_{pi}^{(s)} v_j \overline{(Q_{ij}^{(t)})}' \overline{(Q_{pi}^{(s)})}'], \quad p < i,$$
$$\frac{dv_p}{d\theta} = \sum_s [e_t A_{ip}^{sj} w_{pj}' v_i Q_{ip}^{(s)} + w_{ip}^{(s)} v_j \overline{(Q_{ij}^{(t)})}' Q_{ip}^{(s)}], \quad i < p < j,$$
$$\frac{dv_p}{d\theta} = \sum_s [w_{jp} (A_{ij}^{tp})' e_s' v_i Q_{ip}^{(s)} + w_{ip}^{(s)} v_j \overline{(Q_{ij}^{(t)})}' Q_{ip}^{(s)}], \quad p > j,$$
$$\frac{dw_{ij}}{d\theta} = (r_i r_j - w_{ij} w_{ij}') e_t + 2w_{ij}^{(t)} w_{ij} + 2e_t,$$
$$\frac{dw_{pi}}{d\theta} = r_i \sum_s w_{pj} A_{pi}^{sj} e_t' e_s + \sum_{s,r} w_{pi}^{(s)} w_{ij} (A_{pi}^{rj})' A_{pi}^{sj} e_t' e_r, \quad p < i,$$
$$\frac{dw_{ip}}{d\theta} = r_i \sum_s e_t A_{ip}^{sj} w_{pj}' e_s + \sum_{s,r} w_{ip}^{(s)} w_{ij} A_{ip}^{rj} (A_{ip}^{sj})' e_t' e_r, \quad i < p < j,$$
$$\frac{dw_{ip}}{d\theta} = r_i w_{jp} (A_{ij}^{tp})' + \sum_s w_{ij}^{(s)} w_{ip} A_{ij}^{tp} (A_{ij}^{sp})', \quad p > j,$$

$$\frac{dw_{pj}}{d\theta} = r_j \sum_s w_{pi}^{(s)} e_t (A_{pi}^{sj})' + \sum_s w_{pj} A_{pi}^{sj} e_t' w_{ij} (A_{pi}^{sj})', \quad p < i,$$

$$\frac{dw_{pj}}{d\theta} = r_j \sum_s w_{ip}^{(s)} e_t A_{ip}^{sj} + \sum_s e_t A_{ip}^{sj} w_{pj}' w_{ij} A_{ip}^{sj}, \quad i < p < j,$$

$$\frac{dw_{jp}}{d\theta} = r_j w_{ip} A_{ij}^{tp} + \sum_s w_{ij}^{(s)} w_{jp} (A_{ij}^{tp})' A_{ij}^{sp}, \quad p > j,$$

$$\frac{dw_{pk}}{d\theta} = \sum_{s,r} [w_{pj} A_{pi}^{sj} e_t' e_s A_{pk}^{ri} w_{ki}' e_r + w_{kj} A_{ki}^{sj} e_t' e_s (A_{pk}^{ri})' w_{pi}' e_r], \quad p < k < i,$$

$$\frac{dw_{pk}}{d\theta} = \sum_{s,r} w_{pi}^{(r)} e_t A_{ik}^{sj} w_{kj}' e_s (A_{pi}^{rk})' + \sum_s w_{pj} A_{pi}^{sj} e_t' w_{ik} (A_{pi}^{sk})', \quad p < i < k < j,$$

$$\frac{dw_{pk}}{d\theta} = \sum_s [w_{pi}^{(s)} w_{jk} (A_{ij}^{tk})' (A_{pi}^{sk})' + w_{pj} A_{pi}^{sj} e_t' w_{ik} (A_{pi}^{sk})'], \quad p < i < j < k,$$

$$\frac{dw_{pk}}{d\theta} = \sum_{s,r} w_{ip}^{(r)} e_t A_{ik}^{sj} w_{kj}' e_s A_{ip}^{rk} + \sum_s e_t A_{ip}^{sj} w_{pj}' w_{ik} A_{ip}^{sk}, \quad i < p < k < j,$$

$$\frac{dw_{pk}}{d\theta} = \sum_s [w_{ip}^{(s)} w_{jk} (A_{ij}^{tk})' A_{ip}^{sk} + e_t A_{ip}^{sj} w_{pj}' w_{ik} A_{ip}^{sk}], \quad i < p < j < k,$$

$$\frac{dw_{pk}}{d\theta} = \sum_s [w_{jp} (A_{ij}^{tp})' e_s' w_{ik} A_{ip}^{sk} + w_{ip}^{(s)} w_{jk} (A_{ij}^{tk})' A_{ip}^{sk}], \quad j < p < k.$$

下面求解这个常微分方程组.

(1) 求 r_i, r_j, w_{ij}.

首先计算 $w_{ij}^{(t)}$.

由

$$\frac{dw_{ij}}{d\theta} = (r_i r_j - w_{ij} w_{ij}') e_t + 2 w_{ij}^{(t)} w_{ij} + 2 e_t,$$

得

$$\frac{dw_{ij}^{(t)}}{d\theta} = r_i r_j - \sum_{s \neq t} {w_{ij}^{(s)}}^2 + {w_{ij}^{(t)}}^2 + 2,$$

$$\begin{aligned}
\frac{d^2 w_{ij}^{(t)}}{d\theta^2} &= 4 w_{ij}^{(t)} (r_i r_j - \sum_{s \neq t} {w_{ij}^{(s)}}^2) + 2 w_{ij}^{(t)} \frac{dw_{ij}^{(t)}}{d\theta} \\
&= 4 w_{ij}^{(t)} (\frac{dw_{ij}^{(t)}}{d\theta} - {w_{ij}^{(t)}}^2 - 2) + 2 w_{ij}^{(t)} \frac{dw_{ij}^{(t)}}{d\theta} \\
&= 6 w_{ij}^{(t)} \frac{dw_{ij}^{(t)}}{d\theta} - 4 {w_{ij}^{(t)}}^3 - 8 w_{ij}^{(t)}.
\end{aligned}$$

所以可列出满足初始条件的微分方程

$$\begin{cases} \frac{d^2w_{ij}^{(t)}}{d\theta^2} = 6w_{ij}^{(t)}\frac{dw_{ij}^{(t)}}{d\theta} - 4{w_{ij}^{(t)}}^3 - 8w_{ij}^{(t)} \\ w_{ij}^{(t)}(0) = z_{ij}^{(t)} \\ \frac{dw_{ij}^{(t)}}{d\theta}(0) = s_is_j - \sum\limits_{s\neq t} {z_{ij}^{(s)}}^2 + {z_{ij}^{(t)}}^2 + 2 \end{cases}$$

令 $p = \frac{dw_{ij}^{(t)}}{d\theta}$, 则

$$\frac{d^2w_{ij}^{(t)}}{d\theta^2} = p\frac{dp}{dw_{ij}^{(t)}} = 6w_{ij}^{(t)}p - 4{w_{ij}^{(t)}}^3 - 8w_{ij}^{(t)},$$

$$pdp = (3p - 2{w_{ij}^{(t)}}^2 - 4)d{w_{ij}^{(t)}}^2.$$

令 $u = {w_{ij}^{(t)}}^2$, 则

$$\frac{dp}{du} = \frac{3p - 2u - 4}{p},$$

令 $P = p$, $U = u + 2$, 则

$$\frac{dP}{dU} = 3 - 2\frac{U}{P},$$

令 $q = \frac{P}{U}$, 则

$$U\frac{dq}{dU} + q = 3 - \frac{2}{q},$$

所以

$$-\frac{1}{q-1}dq + \frac{2}{q-2}dq = -\frac{1}{U}dU,$$

即

$$-\frac{1}{1-q}d(1-q) + \frac{2}{q-2}d(q-2) = -\frac{1}{U}dU,$$

$$\frac{(q-2)^2}{1-q} = \frac{1}{cU},$$

$$q = \frac{4cU - 1 \pm \sqrt{1-4cU}}{2cU},$$

$$P = \frac{4cU - 1 \pm \sqrt{1-4cU}}{2c},$$

$$p = \frac{4c(u+2) - 1 \pm \sqrt{1-4c(u+2)}}{2c},$$

$$\frac{dw_{ij}^{(t)}}{d\theta} = \frac{4c({w_{ij}^{(t)}}^2 + 2) - 1 \pm \sqrt{1 - 4c({w_{ij}^{(t)}}^2 + 2)}}{2c} \qquad (3.4.1).$$

不妨在此处取 "+" 号，因为从后面的计算可知，无论取 "+" 还是取 "-", 最后所得的 $w_{ij}^{(t)}$ 都相同.

$$\frac{2cdw_{ij}^{(t)}}{4c({w_{ij}^{(t)}}^2+2)-1+\sqrt{1-4c({w_{ij}^{(t)}}^2+2)}}=d\theta,$$

令

$$\frac{4c}{1-8c}{w_{ij}^{(t)}}^2=\sin^2\alpha,$$

$$\frac{\sqrt{c}d\alpha}{1-\sqrt{1-8c}\cos\alpha}=d\theta,$$

令

$$t=tg\frac{\alpha}{2},$$

$$\frac{2\sqrt{c}}{1+t^2-\sqrt{1-8c}(1-t^2)}dt=d\theta,$$

所以

$$\frac{1}{\sqrt{2}}\text{arctg}\,\frac{2\sqrt{2c}}{1-\sqrt{1-8c}}t=\theta+c',$$

由 $\frac{4c}{1-8c}{w_{ij}^{(t)}}^2=\sin^2\alpha$, 可得

$$w_{ij}^{(t)}=\sqrt{\frac{1-8c}{c}}\frac{t}{1+t^2},$$

$$t=\frac{\sqrt{\frac{1-8c}{c}}+\sqrt{\frac{1-8c}{c}-4{w_{ij}^{(t)}}^2}}{2w_{ij}^{(t)}},$$

所以

$$\frac{\sqrt{1-8c}+\sqrt{1-8c-4c{w_{ij}^{(t)}}^2}}{w_{ij}^{(t)}}=\frac{1-\sqrt{1-8c}}{\sqrt{2}}tg\sqrt{2}(\theta+c'),$$

进一步写出 $w_{ij}^{(t)}$ 的显式表达式

$$\frac{\sqrt{\frac{1-8c}{c}}+\sqrt{\frac{1-8c}{c}-4{w_{ij}^{(t)}}^2}}{2w_{ij}^{(t)}}=\frac{1-\sqrt{1-8c}}{2\sqrt{2c}}tg\sqrt{2}(\theta+c'),$$

令

$$m=\sqrt{\frac{1-8c}{c}},\quad A=\frac{1-\sqrt{1-8c}}{2\sqrt{2c}},$$

$$\frac{m+\sqrt{m^2-4{w_{ij}^{(t)}}^2}}{2w_{ij}^{(t)}}=Atg\sqrt{2}(\theta+c'),$$

$$\sqrt{m^2-4{w_{ij}^{(t)}}^2}=2Atg\sqrt{2}(\theta+c')w_{ij}^{(t)}-m,$$

令

$$\varphi=\theta+c'$$

所以

$$w_{ij}^{(t)}=\frac{mA\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi} \qquad (3.4.2).$$

由 (3.4.1),(3.4.2) 式及初值条件可得:

$$c=\frac{\sum\limits_{s\neq t}{z_{ij}^{(s)}}^2-s_is_j}{(s_is_j-z_{ij}z_{ij}'-2)^2},\qquad c'=\frac{1}{\sqrt{2}}\text{arctg}\,\sqrt{2}\frac{\sqrt{1-8c}+\sqrt{1-8c-4c{z_{ij}^{(t)}}^2}}{1-\sqrt{1-8c}z_{ij}^{(t)}},$$

$r_i, r_j, w_{ij}^{(s)}(s\neq t)$ 的求法相同，只写出 r_i 的求法.

$$\frac{dr_i}{d\theta}=2r_iw_{ij}^{(t)},$$

令

$$x=Atg\sqrt{2}(\theta+c'),$$

则

$$w_{ij}^{(t)}=\frac{mx}{1+x^2},\quad \frac{dx}{d\theta}=\sqrt{2}\frac{x^2+A^2}{A},$$

$$\frac{dr_i}{r_i}=2w_{ij}^{(t)}d\theta=\frac{mA}{\sqrt{2}(A^2-1)}(\frac{1}{1+x^2}-\frac{1}{A^2+x^2})dx^2=(\frac{1}{A^2+x^2}-\frac{1}{1+x^2})dx^2,$$

$$dlnr_i=dln\frac{A^2+x^2}{1+x^2},$$

所以

$$r_i=c_1\frac{A^2+x^2}{1+x^2}$$

由初值条件 $r_i(0)=s_i$ 及 $x=Atg\sqrt{2}(\theta+c')$ 可得

$$c_1=s_i\frac{1+(Atg\sqrt{2}c')^2}{A^2+(Atg\sqrt{2}c')^2}=s_i\frac{\cos^2\sqrt{2}c'+A^2\sin^2\sqrt{2}c'}{A^2},$$

所以

$$r_i=s_i\frac{\cos^2\sqrt{2}c'+A^2\sin^2\sqrt{2}c'}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi}.$$

在验证解的过程中，可以得到一个重要的关系式：

$$mA = \sqrt{2}(1 - A^2).$$

(2) $w_{pi}(p<i)$, $w_{pj}(p<i)$, $w_{ip}(i<p<j)$, $w_{pj}(i<p<j)$, $w_{ip}(p>j)$, $w_{jp}(p>j)$, v_i, v_j 的求法相同. 下面只写出 $w_{pi}(p<i)$ 的求法.

由

$$\frac{dw_{pi}}{d\theta} = r_i \sum_s w_{pj} A_{pi}^{sj} e_t' e_s + \sum_{s,r} w_{pi}^{(s)} w_{ij} (A_{pi}^{rj})' A_{pi}^{sj} e_t' e_r,$$

得

$$\begin{aligned}
\frac{d^2 w_{pi}}{d\theta^2} &= \frac{dr_i}{d\theta} \sum_s w_{pj} A_{pi}^{sj} e_t' e_s + r_i \sum_s \frac{dw_{pj}}{d\theta} A_{pi}^{sj} e_t' e_s \\
&\quad + \sum_{s,r} \frac{dw_{pi}^{(s)}}{d\theta} w_{ij} (A_{pi}^{rj})' A_{pi}^{sj} e_t' e_r + \sum_{s,r} w_{pi}^{(s)} \frac{dw_{ij}}{d\theta} (A_{pi}^{rj})' A_{pi}^{sj} e_t' e_r \\
= {} & 2 r_i w_{ij}^{(t)} \sum_s w_{pj} A_{pi}^{sj} e_t' e_s + r_i r_j \sum_{s,u} w_{pi}^{(u)} e_t (A_{pi}^{uj})' A_{pi}^{sj} e_t' e_s + r_i \sum_{s,u} w_{pj} A_{pi}^{uj} e_t' w_{ij} (A_{pi}^{uj})' A_{pi}^{sj} e_t' e_s \\
&+ r_i \sum_{s,r} w_{pj} A_{pi}^{sj} e_t' w_{ij} (A_{pi}^{rj})' A_{pi}^{sj} e_t' e_r + \sum_{s,r,u} w_{pi}^{(s)} w_{ij} (A_{pi}^{rj})' A_{pi}^{sj} e_t' w_{ij} (A_{pi}^{uj})' A_{pi}^{rj} e_t' e_u \\
&+ 2 w_{ij}^{(t)} \sum_{s,r} w_{pi}^{(s)} w_{ij} (A_{pi}^{rj})' A_{pi}^{sj} e_t' e_r + \sum_{s,r} w_{pi}^{(s)} (r_i r_j - w_{ij} w_{ij}' + 2) e_t (A_{pi}^{rj})' A_{pi}^{sj} e_t' e_r \\
= {} & 4 w_{ij}^{(t)} r_i \sum_s w_{pj} A_{pi}^{sj} e_t' e_s - r_i \sum_{s,u} w_{pj} A_{pi}^{uj} e_t' w_{ij} (A_{pi}^{sj})' A_{pi}^{uj} e_t' e_s \\
&+ r_i \sum_{s,r} w_{pj} A_{pi}^{sj} e_t' w_{ij} (A_{pi}^{rj})' A_{pi}^{sj} e_t' e_r - \sum_{s,r,u} w_{pi}^{(s)} w_{ij} (A_{pi}^{sj})' A_{pi}^{rj} e_t' e_t (A_{pi}^{rj})' A_{pi}^{uj} w_{ij}{}' e_u \\
&+ 2 w_{ij}^{(t)} \sum_{s,u} w_{pi}^{(s)} w_{ij} (A_{pi}^{uj})' A_{pi}^{sj} e_t' e_u + 2 w_{ij}^{(t)} \sum_{s,r} w_{pi}^{(s)} w_{ij} (A_{pi}^{rj})' A_{pi}^{sj} e_t' e_r \\
&+ 2 r_i r_j w_{pi} + (-w_{ij} w_{ij}{}' + 2) w_{pi} \\
= {} & 4 w_{ij}^{(t)} \frac{dw_{pi}}{d\theta} + 2(r_i r_j - w_{ij} w_{ij}' + 1) w_{pi}
\end{aligned}$$

以上计算利用了公式

$$\begin{aligned}
& e_t (A_{pi}^{rj})' A_{pi}^{sj} e_t' = \delta_{s,r}; \\
& \sum_u A_{pi}^{uj} e_t' e_t (A_{pi}^{uj})' = I; \\
& (A_{pi}^{uj})' A_{pi}^{sj} + (A_{pi}^{sj})' A_{pi}^{uj} = 2\delta_{s,u} I^{(n_{ij})}.
\end{aligned}$$

下面求解二阶微分方程：

$$\frac{d^2 w_{pi}}{d\theta^2} = 4 w_{ij}^{(t)} \frac{dw_{pi}}{d\theta} + 2(r_i r_j - w_{ij} w_{ij}{}' + 1) w_{pi}.$$

由 $w_{ij}^{(t)} = \frac{mx}{1+x^2}$, $x = Atg\sqrt{2}(\theta + c')$, $mA = \sqrt{2}(1 - A^2)$, 及

$$\frac{dw_{ij}^{(t)}}{d\theta} = (r_i r_j - w_{ij}w'_{ij} + 2) + 2{w_{ij}^{(t)}}^2,$$

可得

$$\begin{aligned} r_i r_j - w_{ij}w'_{ij} + 1 &= \frac{dw_{ij}^{(t)}}{d\theta} - 2{w_{ij}^{(t)}}^2 - 1 \\ &= m\frac{1-x^2}{(1+x^2)^2}\frac{\sqrt{2}(A^2+x^2)}{A} - 2\frac{m^2x^2}{(1+x^2)^2} - 1 \\ &= \sqrt{2}(1-A^2)\frac{1-x^2}{(1+x^2)^2}\frac{\sqrt{2}(A^2+x^2)}{A^2} - 2\frac{2(1-A^2)^2}{A^2}\frac{x^2}{(1+x^2)^2} - 1 \\ &= \frac{A^2 - 2A^4 + (A^2-2)x^2}{A^2(1+x^2)} \end{aligned}$$

所以

$$\frac{d^2w_{pi}}{d\theta^2} = 4\frac{\sqrt{2}(1-A^2)x}{A(1+x^2)}\frac{dw_{pi}}{d\theta} + 2\frac{A^2-2A^4+(A^2-2)x^2}{A^2(1+x^2)}w_{pi}.$$

令

$$\varphi = \theta + c',$$

$$\begin{aligned} \frac{d^2w_{pi}}{d\varphi^2} =& \frac{4\sqrt{2}(1-A^2)tg\sqrt{2}\varphi}{1+A^2tg^2\sqrt{2}\varphi}\frac{dw_{pi}}{d\varphi} + 2\frac{A^2-2A^4+(A^2-2)A^2tg^2\sqrt{2}\varphi}{A^2(1+A^2tg^2\sqrt{2}\varphi)}w_{pi}, \\ =& \frac{4\sqrt{2}(1-A^2)\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi + A^2\sin^2\sqrt{2}\varphi}\frac{dw_{pi}}{d\varphi} \\ &+ 2\frac{(A^2-2A^4)\cos^2\sqrt{2}\varphi + (A^2-2)A^2\sin^2\sqrt{2}\varphi}{A^2(\cos^2\sqrt{2}\varphi + A^2\sin^2\sqrt{2}\varphi)}w_{pi}, \end{aligned}$$

$$\begin{aligned} &(\cos^2\sqrt{2}\varphi + A^2\sin^2\sqrt{2}\varphi)\frac{d^2w_{pi}}{d\varphi^2} \\ =&4\sqrt{2}(1-A^2)\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi\frac{dw_{pi}}{d\varphi} + [2(1-2A^2)\cos^2\sqrt{2}\varphi + 2(A^2-2)\sin^2\sqrt{2}\varphi]w_{pi}, \\ &((\cos^2\sqrt{2}\varphi + A^2\sin^2\sqrt{2}\varphi)\frac{dw_{pi}}{d\varphi})' \\ =&(2\sqrt{2}(1-A^2)\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi w_{pi})' - 2(\cos^2\sqrt{2}\varphi + A^2\sin^2\sqrt{2}\varphi)w_{pi}, \\ &((\cos^2\sqrt{2}\varphi + A^2\sin^2\sqrt{2}\varphi)w_{pi})'' = -2(\cos^2\sqrt{2}\varphi + A^2\sin^2\sqrt{2}\varphi)w_{pi}. \end{aligned}$$

而 $y'' = -2y$ 的解为： $y = d\sin\sqrt{2}(\varphi + d')$, d,d' 为待定参数.

所以

$$(\cos^2\sqrt{2}\varphi + A^2\sin^2\sqrt{2}\varphi)w_{pi}^{(u)} = d_{pi}^{(u)}\sin\sqrt{2}(\varphi + {d_{pi}^{(u)}}'),$$

$$w_{pi}^{(u)}(\varphi)=\frac{d_{pi}^{(u)}\sin\sqrt{2}(\varphi+d_{pi}^{(u)'})}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi},\quad u=1,\cdots,n_{pi}.$$

下面利用初值条件确定 $d_{pi}^{(u)}$ 和 $d_{pi}^{(u)'}$.

$$\frac{dw_{pi}^{(u)}}{d\varphi}=\frac{\sqrt{2}d_{pi}^{(u)}\cos\sqrt{2}(\varphi+d_{pi}^{(u)'})(\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi)+\sqrt{2}d_{pi}^{(u)}(1-A^2)\sin 2\sqrt{2}\varphi\sin\sqrt{2}(\varphi+d_{pi}^{(u)'})}{(\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi)^2}.$$

初值条件转化为:

$$\begin{cases} w_{pi}^{(u)}(c')=\frac{d_{pi}^{(u)}\sin\sqrt{2}(c'+d_{pi}^{(u)'})}{\cos^2\sqrt{2}c'+A^2\sin^2\sqrt{2}c'}=z_{pi}^{(u)} \\ \frac{dw_{pi}^{(u)}}{d\varphi}(c') \\ =\frac{\sqrt{2}d_{pi}^{(u)}\cos\sqrt{2}(c'+d_{pi}^{(u)'})(\cos^2\sqrt{2}c'+A^2\sin^2\sqrt{2}c')+\sqrt{2}d_{pi}^{(u)}(1-A^2)\sin 2\sqrt{2}c'\sin\sqrt{2}(c'+d_{pi}^{(u)'})}{(\cos^2\sqrt{2}c'+A^2\sin^2\sqrt{2}c')^2} \\ =[s_i\sum_s z_{pj}A_{pi}^{sj}e_t'e_s+\sum_{s,r}z_{pi}^{(s)}z_{ij}(A_{pi}^{rj})'A_{pi}^{sj}e_t'e_r]e_u' \end{cases}$$

令 $D_{pi}^{(u)}=w_{pi}^{(u)}|_{\varphi=c'}$, $T_{pi}^{(u)}=\frac{dw_{pi}^{(u)}}{d\varphi}|_{\varphi=c'}$, 可得

$$d_{pi}^{(u)'}=\operatorname{arcctg}\left(\frac{T_{pi}^{(u)}}{\sqrt{2}D_{pi}^{(u)}}-\frac{(1-A^2)\sin 2\sqrt{2}c'}{\cos^2\sqrt{2}c'+A^2\sin^2\sqrt{2}c'}\right)-c',$$

$$d_{pi}^{(u)}=\frac{D_{pi}^{(u)}(\cos^2\sqrt{2}c'+A^2\sin^2\sqrt{2}c')}{\sin\sqrt{2}(c'+d_{pi}^{(u)'})}.$$

所以

$$w_{pi}^{(u)}(\varphi)=z_{pi}^{(u)}\frac{\sin\sqrt{2}(\varphi+d_{pi}^{(u)'})}{\sin\sqrt{2}(c'+d_{pi}^{(u)'})}\frac{\cos^2\sqrt{2}c'+A^2\sin^2\sqrt{2}c'}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi},\quad u=1,\cdots,n_{pi},$$

其中

$$d_{pi}^{(u)'}=\operatorname{arcctg}\left(\frac{[s_i\sum_s z_{pj}A_{pi}^{sj}e_t'e_s+\sum_{s,r}z_{pi}^{(s)}z_{ij}(A_{pi}^{rj})'A_{pi}^{sj}e_t'e_r]e_u'}{\sqrt{2}z_{pi}^{(u)}}-\frac{(1-A^2)\sin 2\sqrt{2}c'}{\cos^2\sqrt{2}c'+A^2\sin^2\sqrt{2}c'}\right)-c'.$$

(3) $r_p(p<i)$, $r_p(i<p<j)$, $r_p(p>j)$, $w_{pk}(p<k<i)$, $w_{pk}(p<i<k<j)$, $w_{pk}(p<i<j<k)$, $w_{pk}(i<p<k<j)$, $w_{pk}(i<p<j<k)$, $w_{pk}(j<p<k)$, $v_p(p<i)$, $v_p(i<p<j)$, $v_p(p>j)$ 的求法相同. 下面只写出 $r_p(p<i)$ 的求法.

$$\frac{dr_p}{d\theta}=2\sum_s w_{pi}^{(s)}w_{pj}A_{pi}^{sj}e_t',$$

将已求得的 $w_{pi}^{(s)}, w_{pj}$ 代入得

$$\frac{dr_p}{d\varphi} = 2\sum_{s,r} \frac{d_{pi}^{(s)} d_{pj}^{(r)} \sin\sqrt{2}(\varphi + d_{pi}^{(s)'}) \sin\sqrt{2}(\varphi + d_{pj}^{(r)'})}{(\cos^2\sqrt{2}\varphi + A^2 \sin^2\sqrt{2}\varphi)^2} e_r A_{pi}^{(sj)} e_t',$$

$$\begin{aligned}
&\sin\sqrt{2}(\varphi + d_{pi}^{(s)'}) \sin\sqrt{2}(\varphi + d_{pj}^{(r)'}) \\
= \; &\sin^2\sqrt{2}\varphi \cos\sqrt{2}d_{pi}^{(s)'} \cos\sqrt{2}d_{pj}^{(r)'} + \sin\sqrt{2}\varphi \cos\sqrt{2}\varphi \cos\sqrt{2}d_{pi}^{(s)'} \sin\sqrt{2}d_{pj}^{(r)'} \\
&+ \cos\sqrt{2}\varphi \sin\sqrt{2}\varphi \sin\sqrt{2}d_{pi}^{(s)'} \cos\sqrt{2}d_{pj}^{(r)'} + \cos^2\sqrt{2}\varphi \sin\sqrt{2}d_{pi}^{(s)'} \sin\sqrt{2}d_{pj}^{(r)'}
\end{aligned}$$

微分方程的求解转化为求下面的 3 个不定积分:

$$\begin{aligned}
&\text{(I)} \quad \int \frac{\sin^2\sqrt{2}\varphi}{(\cos^2\sqrt{2}\varphi + A^2\sin^2\sqrt{2}\varphi)^2} d\varphi, \\
&\text{(II)} \quad \int \frac{\sin\sqrt{2}\varphi \cos\sqrt{2}\varphi}{(\cos^2\sqrt{2}\varphi + A^2\sin^2\sqrt{2}\varphi)^2} d\varphi, \\
&\text{(III)} \quad \int \frac{\cos^2\sqrt{2}\varphi}{(\cos^2\sqrt{2}\varphi + A^2\sin^2\sqrt{2}\varphi)^2} d\varphi.
\end{aligned}$$

令 $t = tg\sqrt{2}\varphi$, 可得

$$\begin{aligned}
\text{(I)} &= \frac{1}{2\sqrt{2}A^3}(\mathrm{arctg}\, At - \frac{At}{1 + A^2t^2}) + C, \\
\text{(II)} &= \frac{1}{2\sqrt{2}(1 - A^2)} \frac{1}{\cos^2\sqrt{2}\varphi + A^2\sin^2\sqrt{2}\varphi} + C, \\
\text{(III)} &= \frac{1}{2\sqrt{2}A}(\mathrm{arctg}\, At + \frac{At}{1 + A^2t^2}) + C.
\end{aligned}$$

所以

$$\begin{aligned}
r_p(\varphi) = \frac{1}{\sqrt{2}} \sum_{s,r} [&\frac{1}{A^3}(\mathrm{arctg}\, Atg\sqrt{2}\varphi - \frac{Atg\sqrt{2}\varphi}{1 + A^2tg^2\sqrt{2}\varphi}) \cos\sqrt{2}d_{pi}^{(s)'} \cos\sqrt{2}d_{pj}^{(r)'} \\
&+ \frac{1}{1 - A^2} \frac{1}{\cos^2\sqrt{2}\varphi + A^2\sin^2\sqrt{2}\varphi} \sin\sqrt{2}(d_{pi}^{(s)'} + d_{pj}^{(r)'}) \\
&+ \frac{1}{A}(\mathrm{arctg}\, Atg\sqrt{2}\varphi + \frac{Atg\sqrt{2}\varphi}{1 + A^2tg^2\sqrt{2}\varphi}) \sin\sqrt{2}d_{pi}^{(s)'} \sin\sqrt{2}d_{pj}^{(r)'}] d_{pi}^{(s)} d_{pj}^{(r)} e_r A_{pi}^{sj} e_t' + a_0
\end{aligned}$$

由 $r_p(c') = s_p$, 可求出 a_0.

综上， $\exp(\theta(T_{ij}^{(t)}+2\frac{\partial}{\partial z_{ij}^{(t)}}))$ 可表示为：

$$r_i = s_i\frac{\cos^2\sqrt{2}c'+A^2\sin^2\sqrt{2}c'}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi},$$

$$r_j = s_j\frac{\cos^2\sqrt{2}c'+A^2\sin^2\sqrt{2}c'}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi},$$

$$w_{ij} = w_{ij}^{(t)}e_t+\sum_{s\neq t}w_{ij}^{(s)}e_s = \frac{\sqrt{2}(1-A^2)\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi}+\sum_{s\neq t}z_{ij}^{(s)}\frac{\cos^2\sqrt{2}c'+A^2\sin^2\sqrt{2}c'}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi}$$

$$w_{pi}^{(u)} = z_{pi}^{(u)}\frac{\cos^2\sqrt{2}c'+A^2\sin^2\sqrt{2}c'}{\sin\sqrt{2}(c'+d_{pi}^{(u)'})}\frac{\sin\sqrt{2}(\varphi+d_{pi}^{(u)'})}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi},\quad u=1,\cdots,n_{pi},\quad p<i,$$

$$w_{pj}^{(u)} = z_{pj}^{(u)}\frac{\cos^2\sqrt{2}c'+A^2\sin^2\sqrt{2}c'}{\sin\sqrt{2}(c'+d_{pj}^{(u)'})}\frac{\sin\sqrt{2}(\varphi+d_{pj}^{(u)'})}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi},\quad u=1,\cdots,n_{pj},\quad p<i,$$

$$w_{ip}^{(u)} = z_{ip}^{(u)}\frac{\cos^2\sqrt{2}c'+A^2\sin^2\sqrt{2}c'}{\sin\sqrt{2}(c'+d_{ip}^{(u)'})}\frac{\sin\sqrt{2}(\varphi+d_{ip}^{(u)'})}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi},\quad u=1,\cdots,n_{ip},\quad i<p<j,$$

$$w_{pj}^{(u)} = z_{pj}^{(u)}\frac{\cos^2\sqrt{2}c'+A^2\sin^2\sqrt{2}c'}{\sin\sqrt{2}(c'+d_{pj}^{(u)'})}\frac{\sin\sqrt{2}(\varphi+d_{pj}^{(u)'})}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi},\quad u=1,\cdots,n_{pj},\quad i<p<j,$$

$$w_{ip}^{(u)} = z_{ip}^{(u)}\frac{\cos^2\sqrt{2}c'+A^2\sin^2\sqrt{2}c'}{\sin\sqrt{2}(c'+d_{ip}^{(u)'})}\frac{\sin\sqrt{2}(\varphi+d_{ip}^{(u)'})}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi},\quad u=1,\cdots,n_{ip},\quad p>j,$$

$$w_{jp}^{(u)} = z_{jp}^{(u)}\frac{\cos^2\sqrt{2}c'+A^2\sin^2\sqrt{2}c'}{\sin\sqrt{2}(c'+d_{jp}^{(u)'})}\frac{\sin\sqrt{2}(\varphi+d_{jp}^{(u)'})}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi},\quad u=1,\cdots,n_{jp},\quad p>j,$$

$$v_i^{(u)} = d_i^{(u)}\frac{\cos^2\sqrt{2}c'+A^2\sin^2\sqrt{2}c'}{\sin\sqrt{2}(c'+d_i^{(u)'})}\frac{\sin\sqrt{2}(\varphi+d_i^{(u)'})}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi},\quad u=1,\cdots,m_i,$$

$$v_j^{(u)} = d_j^{(u)}\frac{\cos^2\sqrt{2}c'+A^2\sin^2\sqrt{2}c'}{\sin\sqrt{2}(c'+d_j^{(u)'})}\frac{\sin\sqrt{2}(\varphi+d_j^{(u)'})}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi},\quad u=1,\cdots,m_j,$$

$$r_p = g_0(\varphi)+a_0,\quad p<i,$$
$$r_p = g_1(\varphi)+a_1,\quad i<p<j,$$
$$r_p = g_2(\varphi)+a_2,\quad p>j,$$
$$w_{pk} = g_3(\varphi)+a_3,\quad p<k<i,$$
$$w_{pk} = g_4(\varphi)+a_4,\quad p<i<k<j,$$
$$w_{pk} = g_5(\varphi)+a_5,\quad p<i<j<k,$$
$$w_{pk} = g_6(\varphi)+a_6,\quad i<p<k<j,$$
$$w_{pk} = g_7(\varphi)+a_7,\quad i<p<j<k,$$

$$w_{pk} = g_8(\varphi) + a_8, \quad j < p < k,$$
$$v_p = g_9(\varphi) + a_9, \quad p < i,$$
$$v_p = g_{10}(\varphi) + a_{10}, \quad i < p < j,$$
$$v_p = g_{11}(\varphi) + a_{11}, \quad p > j.$$

其中

$$c = \frac{\sum_{s \neq t} {z_{ij}^{(s)}}^2 - s_i s_j}{(s_i s_j - z_{ij} {z_{ij}}' - 2)^2},$$

$$c' = \frac{1}{\sqrt{2}} \operatorname{arctg} \sqrt{2} \frac{\sqrt{1-8c} + \sqrt{1 - 8c - 4c {z_{ij}^{(t)}}^2}}{1 - \sqrt{1-8c} z_{ij}^{(t)}}, \quad A = \frac{1 - \sqrt{1-8c}}{2\sqrt{2c}},$$

$$\varphi = \theta + c',$$

在 $w_{pi}^{(u)}(p < i)$ 的表达式中,

$$d_{pi}^{(u)'} = \operatorname{arcctg}\left(\frac{[s_i \sum_s z_{pj} A_{pi}^{sj} e_t' e_s + \sum_{s,r} z_{pi}^{(s)} z_{ij} (A_{pi}^{rj})' A_{pi}^{sj} e_t' e_r] e_u'}{\sqrt{2} z_{pi}^{(u)}} - \frac{(1 - A^2) \sin 2\sqrt{2}c'}{\cos^2 \sqrt{2}c' + A^2 \sin^2 \sqrt{2}c'}\right) - c';$$

在 $w_{pj}^{(u)}(p < i)$ 的表达式中,

$$d_{pj}^{(u)'} = \operatorname{arcctg}\left(\frac{[s_j \sum_s z_{pi}^{(s)} e_t (A_{pi}^{sj})' + \sum_s z_{pj} A_{pi}^{sj} e_t' z_{ij} (A_{pi}^{sj})'] e_u'}{\sqrt{2} z_{pj}^{(u)}} - \frac{(1 - A^2) \sin 2\sqrt{2}c'}{\cos^2 \sqrt{2}c' + A^2 \sin^2 \sqrt{2}c'}\right) - c';$$

在 $w_{ip}^{(u)}(i < p < j)$ 的表达式中,

$$d_{ip}^{(u)'} = \operatorname{arcctg}\left(\frac{[s_i \sum_s e_t A_{ip}^{sj} z_{pj}' e_s + \sum_{s,r} z_{ip}^{(s)} z_{ij} A_{ip}^{rj} (A_{ip}^{sj})' e_t' e_r] e_u'}{\sqrt{2} z_{ip}^{(u)}} - \frac{(1 - A^2) \sin 2\sqrt{2}c'}{\cos^2 \sqrt{2}c' + A^2 \sin^2 \sqrt{2}c'}\right) - c';$$

在 $w_{pj}^{(u)}(i < p < j)$ 的表达式中,

$$d_{pj}^{(u)'} = \operatorname{arcctg}\left(\frac{[s_j \sum_s z_{ip}^{(s)} e_t A_{ip}^{sj} + \sum_s e_t A_{ip}^{sj} z_{pj}' z_{ij} A_{ip}^{sj}] e_u'}{\sqrt{2} z_{pj}^{(u)}} - \frac{(1 - A^2) \sin 2\sqrt{2}c'}{\cos^2 \sqrt{2}c' + A^2 \sin^2 \sqrt{2}c'}\right) - c';$$

在 $w_{ip}^{(u)}(p > j)$ 的表达式中,

$$d_{ip}^{(u)'} = \operatorname{arcctg}\left(\frac{[s_i z_{jp} (A_{ij}^{tp})' + \sum_s z_{ij}^{(s)} z_{ip} A_{ij}^{tp} (A_{ij}^{sp})'] e_u'}{\sqrt{2} z_{ip}^{(u)}} - \frac{(1 - A^2) \sin 2\sqrt{2}c'}{\cos^2 \sqrt{2}c' + A^2 \sin^2 \sqrt{2}c'}\right) - c';$$

在 $w_{jp}^{(u)}(p > j)$ 的表达式中,

$$d_{jp}^{(u)'} = \operatorname{arcctg}\left(\frac{[s_j z_{ip} A_{ij}^{tp} + \sum_s z_{ij}^{(s)} z_{jp} (A_{ij}^{tp})' A_{ij}^{sp}] e_u'}{\sqrt{2} z_{jp}^{(u)}} - \frac{(1 - A^2) \sin 2\sqrt{2}c'}{\cos^2 \sqrt{2}c' + A^2 \sin^2 \sqrt{2}c'}\right) - c';$$

在 $v_i^{(u)}$ 的表达式中，

$$d_i^{(u)\prime} = \operatorname{arcctg}\left(\frac{[s_i u_j \overline{(Q_{ij}^{(t)})}' + \sum_s z_{ij}^{(s)} u_i Q_{ij}^{(t)} \overline{(Q_{ij}^{(s)})}']e_u'}{\sqrt{2}u_i^{(u)}} - \frac{(1-A^2)\sin 2\sqrt{2}c'}{\cos^2\sqrt{2}c' + A^2\sin^2\sqrt{2}c'}\right) - c';$$

在 $v_j^{(u)}$ 的表达式中，

$$d_j^{(u)\prime} = \operatorname{arcctg}\left(\frac{[s_j u_i Q_{ij}^{(t)} + \sum_s z_{ij}^{(s)} u_j \overline{(Q_{ij}^{(t)})}' Q_{ij}^{(s)}]e_u'}{\sqrt{2}u_j^{(u)}} - \frac{(1-A^2)\sin 2\sqrt{2}c'}{\cos^2\sqrt{2}c' + A^2\sin^2\sqrt{2}c'}\right) - c';$$

在 $r_p(p<i)$ 的表达式中，

$$\begin{aligned} g_0(\varphi) = \frac{1}{\sqrt{2}}\sum_{s,r}[&\frac{\cos\sqrt{2}d_{pi}^{(s)\prime}\cos\sqrt{2}d_{pj}^{(r)\prime}}{A^3}(\operatorname{arctg} A\operatorname{tg}\sqrt{2}\varphi - \frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi + A^2\sin^2\sqrt{2}\varphi}) \\ &+ \frac{\sin\sqrt{2}(d_{pi}^{(s)\prime} + d_{pj}^{(r)\prime})}{1-A^2}\frac{1}{\cos^2\sqrt{2}\varphi + A^2\sin^2\sqrt{2}\varphi} + \frac{\sin\sqrt{2}d_{pi}^{(s)\prime}\sin\sqrt{2}d_{pj}^{(r)\prime}}{A}(\operatorname{arctg} A\operatorname{tg}\sqrt{2}\varphi \\ &+ \frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi + A^2\sin^2\sqrt{2}\varphi})]z_{pi}^{(s)}z_{pj}^{(r)}\frac{(\cos^2\sqrt{2}c' + A^2\sin^2\sqrt{2}c')^2}{\sin\sqrt{2}(c' + d_{pi}^{(s)\prime})\sin\sqrt{2}(c' + d_{pj}^{(r)\prime})}e_r A_{pi}^{sj} e_t', \end{aligned}$$

$a_0 = s_p - g_0(c');$

在 $r_p(i<p<j)$ 的表达式中，

$$\begin{aligned} g_1(\varphi) = \frac{1}{\sqrt{2}}\sum_{s,r}[&\frac{\cos\sqrt{2}d_{ip}^{(s)\prime}\cos\sqrt{2}d_{pj}^{(r)\prime}}{A^3}(\operatorname{arctg} A\operatorname{tg}\sqrt{2}\varphi - \frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi + A^2\sin^2\sqrt{2}\varphi}) \\ &+ \frac{\sin\sqrt{2}(d_{ip}^{(s)\prime} + d_{pj}^{(r)\prime})}{1-A^2}\frac{1}{\cos^2\sqrt{2}\varphi + A^2\sin^2\sqrt{2}\varphi} + \frac{\sin\sqrt{2}d_{ip}^{(s)\prime}\sin\sqrt{2}d_{pj}^{(r)\prime}}{A}(\operatorname{arctg} A\operatorname{tg}\sqrt{2}\varphi \\ &+ \frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi + A^2\sin^2\sqrt{2}\varphi})]z_{ip}^{(s)}z_{pj}^{(r)}\frac{(\cos^2\sqrt{2}c' + A^2\sin^2\sqrt{2}c')^2}{\sin\sqrt{2}(c' + d_{ip}^{(s)\prime})\sin\sqrt{2}(c' + d_{pj}^{(r)\prime})}e_r (A_{ip}^{sj})' e_t', \end{aligned}$$

$a_1 = s_p - g_1(c');$

在 $r_p(p>j)$ 的表达式中，

$$\begin{aligned} g_2(\varphi) = \frac{1}{\sqrt{2}}\sum_{s,r}[&\frac{\cos\sqrt{2}d_{ip}^{(s)\prime}\cos\sqrt{2}d_{jp}^{(r)\prime}}{A^3}(\operatorname{arctg} A\operatorname{tg}\sqrt{2}\varphi - \frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi + A^2\sin^2\sqrt{2}\varphi}) \\ &+ \frac{\sin\sqrt{2}(d_{ip}^{(s)\prime} + d_{jp}^{(r)\prime})}{1-A^2}\frac{1}{\cos^2\sqrt{2}\varphi + A^2\sin^2\sqrt{2}\varphi} + \frac{\sin\sqrt{2}d_{ip}^{(s)\prime}\sin\sqrt{2}d_{jp}^{(r)\prime}}{A}(\operatorname{arctg} A\operatorname{tg}\sqrt{2}\varphi \\ &+ \frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi + A^2\sin^2\sqrt{2}\varphi})]z_{ip}^{(s)}z_{jp}^{(r)}\frac{(\cos^2\sqrt{2}c' + A^2\sin^2\sqrt{2}c')^2}{\sin\sqrt{2}(c' + d_{ip}^{(s)\prime})\sin\sqrt{2}(c' + d_{jp}^{(r)\prime})}e_s A_{ij}^{tp} e_r', \end{aligned}$$

$a_2 = s_p - g_2(c');$

在 $w_{pk}(p<k<i)$ 的表达式中,

$$g_3(\varphi)=\frac{1}{2\sqrt{2}}\sum_{s,r,u,v}[\frac{\cos\sqrt{2}d_{pj}^{(u)'}\cos\sqrt{2}d_{ki}^{(v)'}}{A^3}(\operatorname{arctg}A\operatorname{tg}\sqrt{2}\varphi-\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})$$

$$+\frac{\sin\sqrt{2}(d_{pj}^{(u)'}+d_{ki}^{(v)'})}{1-A^2}\frac{1}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi}$$

$$+\frac{\sin\sqrt{2}d_{pj}^{(u)'}\sin\sqrt{2}d_{ki}^{(v)'}}{A}(\operatorname{arctg}A\operatorname{tg}\sqrt{2}\varphi+\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})]$$

$$z_{pj}^{(u)}z_{ki}^{(v)}\frac{(\cos^2\sqrt{2}c'+A^2\sin^2\sqrt{2}c')^2}{\sin\sqrt{2}(c'+d_{pj}^{(u)'})\sin\sqrt{2}(c'+d_{ki}^{(v)'})}e_uA_{pi}^{sj}e_t'e_sA_{pk}^{ri}e_v'e_r$$

$$+\frac{1}{2\sqrt{2}}\sum_{s,r,u,v}[\frac{\cos\sqrt{2}d_{kj}^{(u)'}\cos\sqrt{2}d_{pi}^{(v)'}}{A^3}(\operatorname{arctg}A\operatorname{tg}\sqrt{2}\varphi-\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})$$

$$+\frac{\sin\sqrt{2}(d_{kj}^{(u)'}+d_{pi}^{(v)'})}{1-A^2}\frac{1}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi}$$

$$+\frac{\sin\sqrt{2}d_{kj}^{(u)'}\sin\sqrt{2}d_{pi}^{(v)'}}{A}(\operatorname{arctg}A\operatorname{tg}\sqrt{2}\varphi+\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})]$$

$$z_{kj}^{(u)}z_{pi}^{(v)}\frac{(\cos^2\sqrt{2}c'+A^2\sin^2\sqrt{2}c')^2}{\sin\sqrt{2}(c'+d_{kj}^{(u)'})\sin\sqrt{2}(c'+d_{pi}^{(v)'})}e_uA_{ki}^{sj}e_t'e_s(A_{pk}^{ri})'e_v'e_r,$$

$a_3=z_{pk}-g_3(c')$;

在 $w_{pk}(p<i<k<j)$ 的表达式中,

$$g_4(\varphi)=\frac{1}{2\sqrt{2}}\sum_{s,r,u}[\frac{\cos\sqrt{2}d_{pi}^{(r)'}\cos\sqrt{2}d_{kj}^{(u)'}}{A^3}(\operatorname{arctg}A\operatorname{tg}\sqrt{2}\varphi-\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})$$

$$+\frac{\sin\sqrt{2}(d_{pi}^{(r)'}+d_{kj}^{(u)'})}{1-A^2}\frac{1}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi}+\frac{\sin\sqrt{2}d_{pi}^{(r)'}\sin\sqrt{2}d_{kj}^{(u)'}}{A}(\operatorname{arctg}A\operatorname{tg}\sqrt{2}\varphi$$

$$+\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})]z_{pi}^{(r)}z_{kj}^{(u)}\frac{(\cos^2\sqrt{2}c'+A^2\sin^2\sqrt{2}c')^2}{\sin\sqrt{2}(c'+d_{pi}^{(r)'})\sin\sqrt{2}(c'+d_{kj}^{(u)'})}e_tA_{ik}^{sj}e_u'e_s(A_{pi}^{rk})'$$

$$+\frac{1}{2\sqrt{2}}\sum_{s,r,u}[\frac{\cos\sqrt{2}d_{pj}^{(r)'}\cos\sqrt{2}d_{ik}^{(u)'}}{A^3}(\operatorname{arctg}A\operatorname{tg}\sqrt{2}\varphi-\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})$$

$$+\frac{\sin\sqrt{2}(d_{pj}^{(r)'}+d_{ik}^{(u)'})}{1-A^2}\frac{1}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi}+\frac{\sin\sqrt{2}d_{pj}^{(r)'}\sin\sqrt{2}d_{ik}^{(u)'}}{A}(\operatorname{arctg}A\operatorname{tg}\sqrt{2}\varphi$$

$$+\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})]z_{pj}^{(r)}z_{ik}^{(u)}\frac{(\cos^2\sqrt{2}c'+A^2\sin^2\sqrt{2}c')^2}{\sin\sqrt{2}(c'+d_{pj}^{(r)'})\sin\sqrt{2}(c'+d_{ik}^{(u)'})}e_rA_{pi}^{sj}e_t'e_u(A_{pi}^{sk})',$$

$a_4 = z_{pk} - g_4(c')$;

在 $w_{pk}(p<i<j<k)$ 的表达式中，

$$g_5(\varphi)=\frac{1}{2\sqrt{2}}\sum_{s,r}[\frac{\cos\sqrt{2}d_{pi}^{(s)'}\cos\sqrt{2}d_{jk}^{(r)'}}{A^3}(\operatorname{arctg}A\operatorname{tg}\sqrt{2}\varphi-\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})$$
$$+\frac{\sin\sqrt{2}(d_{pi}^{(s)'}+d_{jk}^{(r)'})}{1-A^2}\frac{1}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi}+\frac{\sin\sqrt{2}d_{pi}^{(s)'}\sin\sqrt{2}d_{jk}^{(r)'}}{A}(\operatorname{arctg}A\operatorname{tg}\sqrt{2}\varphi$$
$$+\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})]z_{pi}^{(s)}z_{jk}^{(r)}\frac{(\cos^2\sqrt{2}c'+A^2\sin^2\sqrt{2}c')^2}{\sin\sqrt{2}(c'+d_{pi}^{(s)'})\sin\sqrt{2}(c'+d_{jk}^{(r)'})}e_r(A_{ij}^{tk})'(A_{pi}^{sk})'$$
$$+\frac{1}{2\sqrt{2}}\sum_{s,r,u}[\frac{\cos\sqrt{2}d_{pj}^{(r)'}\cos\sqrt{2}d_{ik}^{(u)'}}{A^3}(\operatorname{arctg}A\operatorname{tg}\sqrt{2}\varphi-\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})$$
$$+\frac{\sin\sqrt{2}(d_{pj}^{(r)'}+d_{ik}^{(u)'})}{1-A^2}\frac{1}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi}+\frac{\sin\sqrt{2}d_{pj}^{(r)'}\sin\sqrt{2}d_{ik}^{(u)'}}{A}(\operatorname{arctg}A\operatorname{tg}\sqrt{2}\varphi$$
$$+\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})]z_{pj}^{(r)}z_{ik}^{(u)}\frac{(\cos^2\sqrt{2}c'+A^2\sin^2\sqrt{2}c')^2}{\sin\sqrt{2}(c'+d_{pj}^{(r)'})\sin\sqrt{2}(c'+d_{ik}^{(u)'})}e_rA_{pi}^{sj}e_t'e_u(A_{pi}^{sk})',$$

$a_5 = z_{pk} - g_5(c')$;

在 $w_{pk}(i<p<k<j)$ 的表达式中，

$$g_6(\varphi)=\frac{1}{2\sqrt{2}}\sum_{s,r,u}[\frac{\cos\sqrt{2}d_{ip}^{(r)'}\cos\sqrt{2}d_{kj}^{(u)'}}{A^3}(\operatorname{arctg}A\operatorname{tg}\sqrt{2}\varphi-\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})$$
$$+\frac{\sin\sqrt{2}(d_{ip}^{(r)'}+d_{kj}^{(u)'})}{1-A^2}\frac{1}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi}+\frac{\sin\sqrt{2}d_{ip}^{(r)'}\sin\sqrt{2}d_{kj}^{(u)'}}{A}(\operatorname{arctg}A\operatorname{tg}\sqrt{2}\varphi$$
$$+\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})]z_{ip}^{(r)}z_{kj}^{(u)}\frac{(\cos^2\sqrt{2}c'+A^2\sin^2\sqrt{2}c')^2}{\sin\sqrt{2}(c'+d_{ip}^{(r)'})\sin\sqrt{2}(c'+d_{kj}^{(u)'})}e_tA_{ik}^{sj}e_u'e_sA_{ip}^{rk}$$
$$+\frac{1}{2\sqrt{2}}\sum_{s,r,u}[\frac{\cos\sqrt{2}d_{pj}^{(r)'}\cos\sqrt{2}d_{ik}^{(u)'}}{A^3}(\operatorname{arctg}A\operatorname{tg}\sqrt{2}\varphi-\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})$$
$$+\frac{\sin\sqrt{2}(d_{pj}^{(r)'}+d_{ik}^{(u)'})}{1-A^2}\frac{1}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi}+\frac{\sin\sqrt{2}d_{pj}^{(r)'}\sin\sqrt{2}d_{ik}^{(u)'}}{A}(\operatorname{arctg}A\operatorname{tg}\sqrt{2}\varphi$$
$$+\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})]z_{pj}^{(r)}z_{ik}^{(u)}\frac{(\cos^2\sqrt{2}c'+A^2\sin^2\sqrt{2}c')^2}{\sin\sqrt{2}(c'+d_{pj}^{(r)'})\sin\sqrt{2}(c'+d_{ik}^{(u)'})}e_tA_{ip}^{sj}e_r'e_uA_{ip}^{sk},$$

$a_6 = z_{pk} - g_6(c')$;

在 $w_{pk}(i<p<j<k)$ 的表达式中，

$$g_7(\varphi)=\frac{1}{2\sqrt{2}}\sum_{s,r}[\frac{\cos\sqrt{2}d_{ip}^{(s)'}\cos\sqrt{2}d_{jk}^{(r)'}}{A^3}(\text{arctg}\,A\text{tg}\,\sqrt{2}\varphi-\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})$$

$$+\frac{\sin\sqrt{2}(d_{ip}^{(s)'}+d_{jk}^{(r)'})}{1-A^2}\frac{1}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi}+\frac{\sin\sqrt{2}d_{ip}^{(s)'}\sin\sqrt{2}d_{jk}^{(r)'}}{A}(\text{arctg}\,A\text{tg}\,\sqrt{2}\varphi$$

$$+\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})]z_{ip}^{(s)}z_{jk}^{(r)}\frac{(\cos^2\sqrt{2}c'+A^2\sin^2\sqrt{2}c')^2}{\sin\sqrt{2}(c'+d_{ip}^{(s)'})\sin\sqrt{2}(c'+d_{jk}^{(r)'})}e_r(A_{ij}^{tk})'A_{ip}^{sk}$$

$$+\frac{1}{2\sqrt{2}}\sum_{s,r,u}[\frac{\cos\sqrt{2}d_{pj}^{(r)'}\cos\sqrt{2}d_{ik}^{(u)'}}{A^3}(\text{arctg}\,A\text{tg}\,\sqrt{2}\varphi-\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})$$

$$+\frac{\sin\sqrt{2}(d_{pj}^{(r)'}+d_{ik}^{(u)'})}{1-A^2}\frac{1}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi}+\frac{\sin\sqrt{2}d_{pj}^{(r)'}\sin\sqrt{2}d_{ik}^{(u)'}}{A}(\text{arctg}\,A\text{tg}\,\sqrt{2}\varphi$$

$$+\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})]z_{pj}^{(r)}z_{ik}^{(u)}\frac{(\cos^2\sqrt{2}c'+A^2\sin^2\sqrt{2}c')^2}{\sin\sqrt{2}(c'+d_{pj}^{(r)'})\sin\sqrt{2}(c'+d_{ik}^{(u)'})}e_tA_{ip}^{sj}e_r'e_uA_{ip}^{sk},$$

$a_7=z_{pk}-g_7(c');$

在 $w_{pk}(j<p<k)$ 的表达式中，

$$g_8(\varphi)=\frac{1}{2\sqrt{2}}\sum_{s,r,u}[\frac{\cos\sqrt{2}d_{jp}^{(r)'}\cos\sqrt{2}d_{ik}^{(u)'}}{A^3}(\text{arctg}\,A\text{tg}\,\sqrt{2}\varphi-\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})$$

$$+\frac{\sin\sqrt{2}(d_{jp}^{(r)'}+d_{ik}^{(u)'})}{1-A^2}\frac{1}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi}+\frac{\sin\sqrt{2}d_{jp}^{(r)'}\sin\sqrt{2}d_{ik}^{(u)'}}{A}(\text{arctg}\,A\text{tg}\,\sqrt{2}\varphi$$

$$+\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})]z_{jp}^{(r)}z_{ik}^{(u)}\frac{(\cos^2\sqrt{2}c'+A^2\sin^2\sqrt{2}c')^2}{\sin\sqrt{2}(c'+d_{jp}^{(r)'})\sin\sqrt{2}(c'+d_{ik}^{(u)'})}e_r(A_{ij}^{tp})'e_s'e_uA_{ip}^{sk}$$

$$+\frac{1}{2\sqrt{2}}\sum_{s,r}[\frac{\cos\sqrt{2}d_{ip}^{(s)'}\cos\sqrt{2}d_{jk}^{(r)'}}{A^3}(\text{arctg}\,A\text{tg}\,\sqrt{2}\varphi-\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})$$

$$+\frac{\sin\sqrt{2}(d_{ip}^{(s)'}+d_{jk}^{(r)'})}{1-A^2}\frac{1}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi}+\frac{\sin\sqrt{2}d_{ip}^{(s)'}\sin\sqrt{2}d_{jk}^{(r)'}}{A}(\text{arctg}\,A\text{tg}\,\sqrt{2}\varphi$$

$$+\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})]z_{ip}^{(s)}z_{jk}^{(r)}\frac{(\cos^2\sqrt{2}c'+A^2\sin^2\sqrt{2}c')^2}{\sin\sqrt{2}(c'+d_{ip}^{(s)'})\sin\sqrt{2}(c'+d_{jk}^{(r)'})}e_r(A_{ij}^{tk})'A_{ip}^{sk},$$

$a_8=z_{pk}-g_8(c');$

在 $v_p(p<i)$ 的表达式中，

$$
\begin{aligned}
g_9(\varphi) = \frac{1}{2\sqrt{2}}\sum_{s,r,u}[&\frac{\cos\sqrt{2}d_{pj}^{(r)'}\cos\sqrt{2}d_i^{(u)'}}{A^3}(\operatorname{arctg} A\operatorname{tg}\sqrt{2}\varphi - \frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi + A^2\sin^2\sqrt{2}\varphi}) \\
&+ \frac{\sin\sqrt{2}(d_{pj}^{(r)'} + d_i^{(u)'})}{1-A^2}\frac{1}{\cos^2\sqrt{2}\varphi + A^2\sin^2\sqrt{2}\varphi} + \frac{\sin\sqrt{2}d_{pj}^{(r)'}\sin\sqrt{2}d_i^{(u)'}}{A}(\operatorname{arctg} A\operatorname{tg}\sqrt{2}\varphi \\
&+ \frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi + A^2\sin^2\sqrt{2}\varphi})]z_{pj}^{(r)}u_i^{(u)}\frac{(\cos^2\sqrt{2}c' + A^2\sin^2\sqrt{2}c')^2}{\sin\sqrt{2}(c' + d_{pj}^{(r)'})\sin\sqrt{2}(c' + d_i^{(u)'})}e_r A_{pi}^{sj} e_t' e_u \overline{(Q_{pi}^{(s)})}' \\
&+ \frac{1}{2\sqrt{2}}\sum_{s,r}[\frac{\cos\sqrt{2}d_{pi}^{(s)'}\cos\sqrt{2}d_j^{(r)'}}{A^3}(\operatorname{arctg} A\operatorname{tg}\sqrt{2}\varphi - \frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi + A^2\sin^2\sqrt{2}\varphi}) \\
&+ \frac{\sin\sqrt{2}(d_{pi}^{(s)'} + d_j^{(r)'})}{1-A^2}\frac{1}{\cos^2\sqrt{2}\varphi + A^2\sin^2\sqrt{2}\varphi} + \frac{\sin\sqrt{2}d_{pi}^{(s)'}\sin\sqrt{2}d_j^{(r)'}}{A}(\operatorname{arctg} A\operatorname{tg}\sqrt{2}\varphi \\
&+ \frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi + A^2\sin^2\sqrt{2}\varphi})]z_{pi}^{(s)}u_j^{(r)}\frac{(\cos^2\sqrt{2}c' + A^2\sin^2\sqrt{2}c')^2}{\sin\sqrt{2}(c' + d_{pi}^{(s)'})\sin\sqrt{2}(c' + d_j^{(r)'})}e_r \overline{(Q_{ij}^{(t)})}' \overline{(Q_{pi}^{(s)})}',
\end{aligned}
$$

$a_9 = u_p - g_9(c')$;

在 $v_p(i<p<j)$ 的表达式中，

$$
\begin{aligned}
g_{10}(\varphi) = \frac{1}{2\sqrt{2}}\sum_{s,r,u}[&\frac{\cos\sqrt{2}d_{pj}^{(r)'}\cos\sqrt{2}d_i^{(u)'}}{A^3}(\operatorname{arctg} A\operatorname{tg}\sqrt{2}\varphi - \frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi + A^2\sin^2\sqrt{2}\varphi}) \\
&+ \frac{\sin\sqrt{2}(d_{pj}^{(r)'} + d_i^{(u)'})}{1-A^2}\frac{1}{\cos^2\sqrt{2}\varphi + A^2\sin^2\sqrt{2}\varphi} + \frac{\sin\sqrt{2}d_{pj}^{(r)'}\sin\sqrt{2}d_i^{(u)'}}{A}(\operatorname{arctg} A\operatorname{tg}\sqrt{2}\varphi \\
&+ \frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi + A^2\sin^2\sqrt{2}\varphi})]z_{pj}^{(r)}u_i^{(u)}\frac{(\cos^2\sqrt{2}c' + A^2\sin^2\sqrt{2}c')^2}{\sin\sqrt{2}(c' + d_{pj}^{(r)'})\sin\sqrt{2}(c' + d_i^{(u)'})}e_t A_{ip}^{sj} e_r' e_u Q_{ip}^{(s)} \\
&+ \frac{1}{2\sqrt{2}}\sum_{s,r}[\frac{\cos\sqrt{2}d_{ip}^{(s)'}\cos\sqrt{2}d_j^{(r)'}}{A^3}(\operatorname{arctg} A\operatorname{tg}\sqrt{2}\varphi - \frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi + A^2\sin^2\sqrt{2}\varphi}) \\
&+ \frac{\sin\sqrt{2}(d_{ip}^{(s)'} + d_j^{(r)'})}{1-A^2}\frac{1}{\cos^2\sqrt{2}\varphi + A^2\sin^2\sqrt{2}\varphi} + \frac{\sin\sqrt{2}d_{ip}^{(s)'}\sin\sqrt{2}d_j^{(r)'}}{A}(\operatorname{arctg} A\operatorname{tg}\sqrt{2}\varphi \\
&+ \frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi + A^2\sin^2\sqrt{2}\varphi})]z_{ip}^{(s)}u_j^{(r)}\frac{(\cos^2\sqrt{2}c' + A^2\sin^2\sqrt{2}c')^2}{\sin\sqrt{2}(c' + d_{ip}^{(s)'})\sin\sqrt{2}(c' + d_j^{(r)'})}e_r \overline{(Q_{ij}^{(t)})}' Q_{ip}^{(s)},
\end{aligned}
$$

$a_{10} = u_p - g_{10}(c')$;

在 $v_p(p>j)$ 的表达式中,

$$g_{11}(\varphi)=\frac{1}{2\sqrt{2}}\sum_{s,r,u}[\frac{\cos\sqrt{2}d_{jp}^{(r)'}\cos\sqrt{2}d_i^{(u)'}}{A^3}(\operatorname{arctg}A\operatorname{tg}\sqrt{2}\varphi-\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})$$

$$+\frac{\sin\sqrt{2}(d_{jp}^{(r)'}+d_i^{(u)'})}{1-A^2}\frac{1}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi}+\frac{\sin\sqrt{2}d_{jp}^{(r)'}\sin\sqrt{2}d_i^{(u)'}}{A}(\operatorname{arctg}A\operatorname{tg}\sqrt{2}\varphi$$

$$+\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})]z_{jp}^{(r)}u_i^{(u)}\frac{(\cos^2\sqrt{2}c'+A^2\sin^2\sqrt{2}c')^2}{\sin\sqrt{2}(c'+d_{jp}^{(r)'})\sin\sqrt{2}(c'+d_i^{(u)'})}e_r(A_{ij}^{tp})'e_s'e_uQ_{ip}^{(s)}$$

$$+\frac{1}{2\sqrt{2}}\sum_{s,r}[\frac{\cos\sqrt{2}d_{ip}^{(s)'}\cos\sqrt{2}d_j^{(r)'}}{A^3}(\operatorname{arctg}A\operatorname{tg}\sqrt{2}\varphi-\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})$$

$$+\frac{\sin\sqrt{2}(d_{ip}^{(s)'}+d_j^{(r)'})}{1-A^2}\frac{1}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi}+\frac{\sin\sqrt{2}d_{ip}^{(s)'}\sin\sqrt{2}d_j^{(r)'}}{A}(\operatorname{arctg}A\operatorname{tg}\sqrt{2}\varphi$$

$$+\frac{A\sin\sqrt{2}\varphi\cos\sqrt{2}\varphi}{\cos^2\sqrt{2}\varphi+A^2\sin^2\sqrt{2}\varphi})]z_{ip}^{(s)}u_j^{(r)}\frac{(\cos^2\sqrt{2}c'+A^2\sin^2\sqrt{2}c')^2}{\sin\sqrt{2}(c'+d_{ip}^{(s)'})\sin\sqrt{2}(c'+d_j^{(r)'})}e_r\overline{(Q_{ij}^{(t)})'}Q_{ip}^{(s)},$$

$\sigma_{11}=u_p-g_{11}(c')$.

正规 Siegel 域 $D(v_N,F)$ 中固定点 p 的迷向子群 $\operatorname{Iso}(D(v_N,F))$ 由 $\exp(o(D(v_N,F)))$, $\exp(\widetilde{y}(D(v_N,F)))$, $\exp(\widetilde{L_1})$, $\exp(\widetilde{L_2})$ 生成. 而 $\widetilde{y}(D(v_N,F))$ 以 $X_{ij}^{(t)}-Z_{ij}^{(t)}$ 为基, $\widetilde{L_1}$ 以 $Y_i^{(t)}-\widetilde{P}_i^{(t)}$, $\widetilde{Y}_i^{(t)}+P_i^{(t)}$ 为基, $\widetilde{L_2}$ 以 $B_i+\frac{\partial}{\partial s_i}$, $T_{ij}^{(t)}+2\frac{\partial}{\partial z_{ij}^{(t)}}$ 为基. 又 $X_{ij}^{(t)}-Z_{ij}^{(t)}$, $Y_i^{(t)}-\widetilde{P}_i^{(t)}$, $\widetilde{Y}_i^{(t)}+P_i^{(t)}$, $B_i+\frac{\partial}{\partial s_i}$, $T_{ij}^{(t)}+2\frac{\partial}{\partial z_{ij}^{(t)}}$ 所决定的单参数子群分别为 $\exp(\theta(X_{ij}^{(t)}-Z_{ij}^{(t)}))$, $\exp(\theta(Y_i^{(t)}-\widetilde{P}_i^{(t)}))$, $\exp(\theta(\widetilde{Y}_i^{(t)}+P_i^{(t)}))$, $\exp(\operatorname{arctg}\theta(B_i+\frac{\partial}{\partial s_i}))$, $\exp(\theta(T_{ij}^{(t)}+2\frac{\partial}{\partial z_{ij}^{(t)}}))$. 所以, $\exp(\widetilde{y}(D(v_N,F)))$ 以 $\exp(\theta(X_{ij}^{(t)}-Z_{ij}^{(t)}))$ 为生成元, $\exp(\widetilde{L_1})$ 以 $\exp(\theta(Y_i^{(t)}-\widetilde{P}_i^{(t)}))$, $\exp(\theta(\widetilde{Y}_i^{(t)}+P_i^{(t)}))$ 为生成元, $\exp(\widetilde{L_2})$ 以 $\exp(\operatorname{arctg}\theta(B_i+\frac{\partial}{\partial s_i}))$, $\exp(\theta(T_{ij}^{(t)}+2\frac{\partial}{\partial z_{ij}^{(t)}}))$ 为生成元. 至此完成了定理的证明.

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